STEP Support Programme - Cambridge state school offer-holders day
Workshop 2 Questions: Solutions

For all of these questions note that these are **final** solutions, what you cannot see are the crossings out, wrong turns, places where working in two directions meets in the middle etc. There are more explanations and other commentary that you would not be expected to include. You are not expected to produce solutions that look like these in the exams!

1 **2013 S2 Q11**

The key thing here is a system of labelling the velocities which is easy to follow. I have used u_1, u_2, u_3 for the initial velocities, v_1, v_2, v_3 for the velocities after the first collision, w_1, w_2, w_3 for the velocities after the second collision and y_1, y_2, y_3 for the velocities after the third collision.[1]

The starting values are $u_1 = u$, $u_2 = 0$ and $u_3 = 0$. All the masses are the same so we will call them all m.

(i) For the first collision we have:

$$\text{Conservation of momentum:} \quad mu = mv_1 + mv_2 \implies u = v_1 + v_2 \qquad (1)$$
$$\text{Law of restitution:} \quad eu = v_2 - v_1 \qquad (2)$$

Labelling your equations helps to make your method clear. You can use numbers, letters, or things like $(*)$ and $(\dagger)$.

Solving these simultaneously gives:

$$(1) - (2) \implies v_1 = \tfrac{1}{2}u(1-e)$$
$$(1) + (2) \implies v_2 = \tfrac{1}{2}u(1+e)\,.$$

Note that $v_2 > v_1$ which is what we would expect (particle 1 cannot pass through particle 2).[2]

For the second collision (which will be between particles 2 and 3) we have:

$$\text{Conservation of momentum:} \quad mv_2 = mw_2 + mw_3 \implies v_2 = w_2 + w_3$$
$$\text{Law of restitution:} \quad ev_2 = w_3 - w_2$$

Solving these in exactly the same way as before gives:

$$w_2 = \tfrac{1}{2}v_2(1-e)$$
$$w_3 = \tfrac{1}{2}v_2(1+e)$$

and we also have $w_1 = v_1$, as nothing has collided with particle 1 during this collision.

[1] You can use different labelling systems, but whatever you choose it is helpful to define everything clearly. Some of these values will be unchanged at a collision but I found it easier to change all of the labels at each collision.

[2] It is a good idea to check that your answers are sensible at each stage — it means that you can "catch" any mistakes quicker.

Substituting $v_2 = \frac{1}{2}u(1+e)$ into the expressions for w_2 and w_3 gives the velocities after the second collision as:

$$\begin{aligned}
w_1 &= \tfrac{1}{2}u(1-e)\\
w_2 &= \tfrac{1}{2}(1-e) \times \tfrac{1}{2}u(1+e) = \tfrac{1}{4}u(1-e^2)\\
w_3 &= \tfrac{1}{2}(1+e) \times \tfrac{1}{2}u(1+e) = \tfrac{1}{4}u(1+e)^2
\end{aligned}$$

From above, we have $w_3 > w_2$, as makes sense. For a third collision we need to have $w_1 > w_2$. Consider $w_1 - w_2$:[3]

$$\begin{aligned}
w_1 - w_2 &= \tfrac{1}{2}u(1-e) - \tfrac{1}{4}u(1-e^2)\\
&= \tfrac{1}{4}u\left(2(1-e) - (1-e^2)\right)\\
&= \tfrac{1}{4}u\left(1 - 2e + e^2\right)\\
&= \tfrac{1}{4}u(1-e)^2
\end{aligned}$$

and since $e < 1$ (i.e. is not equal to 1 which would mean $w_1 = w_2$) we know that $w_1 - w_2 > 0$ and hence $w_1 > w_2$ and there will be another collision for all values of e where $0 < e < 1$.

(ii) For the third collision we have:

Conservation of momentum:	$mw_1 + mw_2 = my_1 + my_2 \implies w_1 + w_2 = y_1 + y_2$
Law of restitution:	$e(w_1 - w_2) = y_2 - y_1$

Pausing to think for a moment, we want to show that there will be a fourth collision which means we want $y_2 > w_3$. Hence we don't actually need to find y_1!

Solving the equations for y_2 gives:

$$\begin{aligned}
y_2 &= \tfrac{1}{2}(w_1 + w_2 + ew_1 - ew_2)\\
&= \tfrac{1}{2}\left(w_1(1+e) + w_2(1-e)\right)\\
&= \tfrac{1}{2}\left(\tfrac{1}{2}u(1-e)(1+e) + \tfrac{1}{4}u(1-e^2)(1-e)\right)\\
&= \tfrac{1}{8}\left(2u(1-e)(1+e) + u(1-e^2)(1-e)\right)
\end{aligned}$$

[3] If you need to show that $A > B$ it is often easier to show that $A - B > 0$.

We have a fourth collision iff[4] $y_2 - w_3 > 0$, so we want:

$$\begin{aligned} \tfrac{1}{8}\left(2u(1-e)(1+e) + u(1-e^2)(1-e)\right) - \tfrac{1}{4}u(1+e)^2 &> 0 \\ \tfrac{1}{8}u(1+e)\left(2(1-e) + (1-e)^2 - 2(1+e)\right) &> 0^{[5]} \\ \tfrac{1}{8}u(1+e)\left(e^2 - 6e + 1\right) &> 0 \end{aligned}$$

u and $1+e$ are both positive, so we need $e^2 - 6e + 1 > 0$. Solving $e^2 - 6e + 1 = 0$ gives the solutions $e = \dfrac{6 \pm \sqrt{32}}{2} = 3 \pm \sqrt{8}$ and — remembering that $0 < e < 1$ — we can conclude that there will be a fourth collision iff $0 < e < 3 - \sqrt{8}$.

When solving the inequality $e^2 - 6e + 1 > 0$ it may be helpful to do a (very) rough sketch of the quadratic.

This is the sort of question where a stray negative sign can cause havoc. Checking that your answers are sensible at each stage is a good way of catching a mistake before it gets much further. However in an exam situation you need to weigh up the benefit of spending time tracking down the mistake against the benefit of trying another question.

[4] If and only if.

[5] When simplifying expressions it is almost always a good idea to factorise out any common factors first.

2 2013 S2 Q12

Here we need the definitions of $\mathrm{E}(X)$, $\mathrm{Var}\,(X)$, the probabilities of the Poisson distribution $\mathrm{P}(U=r) = \dfrac{\mathrm{e}^{-\lambda}\lambda^r}{r!}$ and the variance of the Poisson distribution, $\mathrm{Var}\,(U) = \lambda$. Everything else is manipulating sums and equations.

(i) We have:

$$\begin{aligned}
\mathrm{E}(X) &= 1 \times \frac{\mathrm{e}^{-\lambda}\lambda^1}{1!} + 3 \times \frac{\mathrm{e}^{-\lambda}\lambda^3}{3!} + 5 \times \frac{\mathrm{e}^{-\lambda}\lambda^5}{5!} + \dots \\
&= \mathrm{e}^{-\lambda}\lambda^1 + \cancel{3} \times \frac{\mathrm{e}^{-\lambda}\lambda^3}{\cancel{3} \times 2!} + \cancel{5} \times \frac{\mathrm{e}^{-\lambda}\lambda^5}{\cancel{5} \times 4!} + \dots \\
&= \mathrm{e}^{-\lambda}\lambda\left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \dots\right) \\
&= \mathrm{e}^{-\lambda}\lambda\alpha\,.
\end{aligned}$$

Similarly:

$$\begin{aligned}
\mathrm{E}(Y) &= 2 \times \frac{\mathrm{e}^{-\lambda}\lambda^2}{2!} + 4 \times \frac{\mathrm{e}^{-\lambda}\lambda^4}{4!} + 6 \times \frac{\mathrm{e}^{-\lambda}\lambda^6}{6!} + \dots \\
&= \mathrm{e}^{-\lambda}\lambda\left(\lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \dots\right) \\
&= \mathrm{e}^{-\lambda}\lambda\beta\,.
\end{aligned}$$

(ii) We have $\mathrm{Var}\,(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$. First find $\mathrm{E}(X^2)$:

$$\begin{aligned}
\mathrm{E}(X^2) &= 1^2 \times \frac{\mathrm{e}^{-\lambda}\lambda^1}{1!} + 3^2 \times \frac{\mathrm{e}^{-\lambda}\lambda^3}{3!} + 5^2 \times \frac{\mathrm{e}^{-\lambda}\lambda^5}{5!} + \dots \\
&= \mathrm{e}^{-\lambda}\lambda^1 + \cancel{3} \times 3 \times \frac{\mathrm{e}^{-\lambda}\lambda^3}{\cancel{3} \times 2!} + \cancel{5} \times 5 \times \frac{\mathrm{e}^{-\lambda}\lambda^5}{\cancel{5} \times 4!} + \dots \\
&= \mathrm{e}^{-\lambda}\lambda\left(1 + \frac{3\lambda^2}{2!} + \frac{5\lambda^4}{4!} + \dots\right) \\
&= \mathrm{e}^{-\lambda}\lambda\left(1 + \frac{(1+2)\lambda^2}{2!} + \frac{(1+4)\lambda^4}{4!} + \dots\right) \\
&= \mathrm{e}^{-\lambda}\lambda\left(1 + \frac{\lambda^2}{2!} + \frac{\cancel{2}\lambda^2}{\cancel{2} \times 1!} + \frac{\lambda^4}{4!} + \frac{\cancel{4}\lambda^4}{\cancel{4} \times 3!} + \dots\right) \\
&= \mathrm{e}^{-\lambda}\lambda\left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \dots + \lambda\left[\frac{\lambda}{1!} + \frac{\lambda^3}{3!} + \dots\right]\right) \\
&= \mathrm{e}^{-\lambda}\lambda\,(\alpha + \lambda\beta)
\end{aligned}$$

Then we have:

$$\begin{aligned}\mathrm{Var}\,(X) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \\ &= \mathrm{e}^{-\lambda}\lambda\,(\alpha + \lambda\beta) - \left(\mathrm{e}^{-\lambda}\lambda\alpha\right)^2\end{aligned}$$

which is not quite the required result. However, we have:

$$\alpha + \beta = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \ldots = \mathrm{e}^{\lambda}$$

and hence $\mathrm{e}^{-\lambda} = \dfrac{1}{\alpha + \beta}$. This gives $\mathrm{Var}\,(X) = \dfrac{\lambda\alpha + \lambda^2\beta}{\alpha + \beta} - \dfrac{\lambda^2\alpha^2}{(\alpha + \beta)^2}$.

The same approach gives $\mathrm{Var}\,(Y) = \dfrac{\lambda\beta + \lambda^2\alpha}{\alpha + \beta} - \dfrac{\lambda^2\beta^2}{(\alpha + \beta)^2}$.

You would need to show some working for this part as well! Perhaps not quite as much as for $\mathrm{Var}(X)$, as that was a "show that", but more than I have done here.

For the last part, start by noting that $\mathrm{Var}\,(X + Y) = \mathrm{Var}\,(U) = \lambda$. We then want to find non-zero values of λ for which:

$$\frac{\lambda\alpha + \lambda^2\beta}{\alpha + \beta} - \frac{\lambda^2\alpha^2}{(\alpha + \beta)^2} + \frac{\lambda\beta + \lambda^2\alpha}{\alpha + \beta} - \frac{\lambda^2\beta^2}{(\alpha + \beta)^2} = \lambda$$

$$\text{i.e.} \qquad \lambda(\alpha + \lambda\beta)(\alpha + \beta) - \lambda^2\alpha^2 + \lambda(\beta + \lambda\alpha)(\alpha + \beta) - \lambda^2\beta^2 = \lambda(\alpha + \beta)^2$$

Then either $\lambda = 0$, or:

$$\begin{aligned}(\alpha + \lambda\beta)(\alpha + \beta) - \lambda\alpha^2 + (\beta + \lambda\alpha)(\alpha + \beta) - \lambda\beta^2 &= (\alpha + \beta)^2 \\ \cancel{\alpha^2} + \cancel{\alpha\beta} + \lambda\alpha\beta + \cancel{\lambda\beta^2} - \cancel{\lambda\alpha^2} + \cancel{\alpha\beta} + \cancel{\beta^2} + \cancel{\lambda\alpha^2} + \lambda\alpha\beta - \cancel{\lambda\beta^2} &= \cancel{\alpha^2} + \cancel{2\alpha\beta} + \cancel{\beta^2} \\ \Longrightarrow \qquad 2\lambda\alpha\beta &= 0\end{aligned}$$

You could use different colours to show which bits cancel out.

If $\lambda \neq 0$ this can only be solved if one of α and β is zero. Since $\alpha > 0$ and $\beta > 0$ there are no non-zero values of λ for which $\mathrm{Var}\,(X) + \mathrm{Var}\,(Y) = \mathrm{Var}\,(X + Y)$.

We know that $\alpha > 0$ as it is equal to $1+$ an infinite number of positive terms, and we know $\beta > 0$ as it is equal to $\lambda \times (1 + \text{an infinite number of positive terms})$, and so can only be equal to 0 when $\lambda = 0$. Note that since λ is the parameter in the Poisson distribution of U we must have $\lambda \geqslant 0$.

3 2013 S3 Q1

Let $t = \tan\left(\frac{x}{2}\right)$. Then:

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}x} &= \frac{1}{2}\sec^2\frac{x}{2}\\ &= \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)\\ &= \frac{1}{2}\left(1+t^2\right)\end{aligned}$$

as required. Using the double-angle formula, $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ which can be written as $2\tan\frac{x}{2}\cos^2\frac{x}{2}$.

Then using $\cos^2\theta = \frac{1}{\sec^2\theta}$ and $\sec^2\theta = 1+\tan^2\theta$ gives $\sin x = 2\times t\times\frac{1}{1+t^2} = \frac{2t}{1+t^2}$ as required.

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\frac{1}{1+a\sin x}\,\mathrm{d}x &= \int_{x=0}^{x=\frac{\pi}{2}}\frac{1}{1+\frac{2at}{1+t^2}}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t\\ &= \int_{x=0}^{x=\frac{\pi}{2}}\frac{\cancel{1+t^2}}{1+t^2+2at}\frac{2}{\cancel{1+t^2}}\,\mathrm{d}t\\ &= 2\int_{t=0}^{t=1}\frac{1}{t^2+2at+1}\,\mathrm{d}t\\ &= 2\int_0^1\frac{1}{(t+a)^2+(1-a^2)}\,\mathrm{d}t\end{aligned}$$

In the "list of required formulae" we have $\displaystyle\int\frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1}x + c$. You are expected to be able to quote this, or derive this quickly and also derive or write down similar results:[6]

Using $\displaystyle\int\frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$ gives:

$$2\left[\frac{1}{\sqrt{1-a^2}}\arctan\frac{t+a}{\sqrt{1-a^2}}\right]_0^1$$

which is

$$\frac{2}{\sqrt{1-a^2}}\left[\arctan\frac{a+1}{\sqrt{1-a^2}} - \arctan\frac{a}{\sqrt{1-a^2}}\right]$$

[6]This question was written when there was a formula book accompanying the papers. The formula book included the formula for $\int\frac{1}{a^2+x^2}\,\mathrm{d}x$. If this question was set now then you might have been given this integral in the question, especially as there is quite a lot of integration to do already.

We can simplify the difference between the two arctans using the formula for the difference between two tans:[7]

$$
\begin{aligned}
&\frac{2}{\sqrt{1-a^2}}\arctan\left(\tan\left[\arctan\frac{a+1}{\sqrt{1-a^2}}-\arctan\frac{a}{\sqrt{1-a^2}}\right]\right)\\
=&\frac{2}{\sqrt{1-a^2}}\arctan\left(\frac{\frac{a+1}{\sqrt{1-a^2}}-\frac{a}{\sqrt{1-a^2}}}{1+\frac{a+1}{\sqrt{1-a^2}}\times\frac{a}{\sqrt{1-a^2}}}\right)\\
=&\frac{2}{\sqrt{1-a^2}}\arctan\left(\frac{\sqrt{1-a^2}\left((\cancel{a}+1)-\cancel{a}\right)}{(1-a^2)+a(a+1)}\right)\\
=&\frac{2}{\sqrt{1-a^2}}\arctan\left(\frac{\sqrt{1-a^2}}{1+a}\right)\\
=&\frac{2}{\sqrt{1-a^2}}\arctan\left(\frac{\sqrt{1-a}}{\sqrt{1+a}}\right)
\end{aligned}
$$

Let

$$I_n=\int_0^{\frac{\pi}{2}}\frac{\sin^n x}{2+\sin x}\,\mathrm{d}x$$

Then

$$
\begin{aligned}
I_{n+1}+2I_n&=\int_0^{\frac{\pi}{2}}\frac{\sin^{n+1}x+2\sin^n x}{2+\sin x}\,\mathrm{d}x\\
&=\int_0^{\frac{\pi}{2}}\sin^n x\left(\frac{\sin x+2}{2+\sin x}\right)\,\mathrm{d}x\\
&=\int_0^{\frac{\pi}{2}}\sin^n x\,\mathrm{d}x
\end{aligned}
$$

Thus

$$I_{n+1}=\int_0^{\frac{\pi}{2}}\sin^n x\,\mathrm{d}x-2I_n$$

Repeated application of this gives

$$
\begin{aligned}
I_3&=\int_0^{\frac{\pi}{2}}\sin^2 x\,\mathrm{d}x-2I_2\\
&=\int_0^{\frac{\pi}{2}}\sin^2 x\,\mathrm{d}x-2\int_0^{\frac{\pi}{2}}\sin x\,\mathrm{d}x+4I_1\\
&=\int_0^{\frac{\pi}{2}}\sin^2 x\,\mathrm{d}x-2\int_0^{\frac{\pi}{2}}\sin x\,\mathrm{d}x+4\int_0^{\frac{\pi}{2}}\sin^0 x\,\mathrm{d}x-8I_0\\
&=\int_0^{\frac{\pi}{2}}\left[\tfrac{1}{2}(1-\cos 2x)-2\sin x+4\right]\mathrm{d}x-8I_0\\
&=\left[\tfrac{9}{2}x-\tfrac{1}{4}\sin 2x+2\cos x\right]_0^{\frac{\pi}{2}}-8I_0\\
&=\tfrac{9}{4}\pi-2-8I_0
\end{aligned}
$$

[7] See Q**9** for more detail on using $\arctan(\tan x)$ and $\tan(\arctan x)$, these are not both necessarily equal to x.

I_0 can be evaluated using the earlier result:

$$\begin{aligned}
I_0 &= \int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x}\,\mathrm{d}x \\
&= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{1}{2}\sin x}\,\mathrm{d}x \\
&= \tfrac{1}{2} \times \frac{2}{\sqrt{1-\frac{1}{4}}} \arctan \frac{\sqrt{1-\frac{1}{2}}}{\sqrt{1+\frac{1}{2}}} \\
&= \frac{2}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}} \\
&= \frac{2}{\sqrt{3}} \times \frac{\pi}{6} = \frac{\sqrt{3}\pi}{9}
\end{aligned}$$

giving a final answer of:

$$\left(\frac{9}{4} - \frac{8\sqrt{3}}{9}\right)\pi - 2$$

.

4 **2012 S3 Q1**

To show the "stem"[8]. result, use the product rule and implicit differentiation to get:

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= ny^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x} \times \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y^n \times 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \\ &= y^{n-1} \times \frac{\mathrm{d}y}{\mathrm{d}x} \times \left(n\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\end{aligned}$$

(i) Comparing the equation to the "stem" result, $n = 1$ might be a good thing to consider. Taking $n = 1$ we have:

$$\begin{aligned} z &= y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \qquad (*) \\ \frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}x}\left(\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\end{aligned}$$

Multiplying the equation given in the question throughout by $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ gives:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}\left(\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) &= \sqrt{y}\,\frac{\mathrm{d}y}{\mathrm{d}x} \\ \frac{\mathrm{d}z}{\mathrm{d}x} &= \sqrt{y}\,\frac{\mathrm{d}y}{\mathrm{d}x} \\ \frac{\mathrm{d}z}{\mathrm{d}x} &= \sqrt{z} \qquad \text{using } (*)\end{aligned}$$

Separating the variables gives:

$$\begin{aligned}\int z^{-\frac{1}{2}}\,\mathrm{d}z &= \int 1\,\mathrm{d}x \\ 2z^{\frac{1}{2}} &= x + c \\ 2y^{\frac{1}{2}}\,\frac{\mathrm{d}y}{\mathrm{d}x} &= x + c\end{aligned}$$

Using the initial conditions $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$ gives $c = 0$. Integrating again gives:

$$\begin{aligned}\int 2y^{\frac{1}{2}}\,\mathrm{d}y &= \int x\,\mathrm{d}x \\ \tfrac{4}{3}y^{\frac{3}{2}} &= \tfrac{1}{2}x^2 + k\end{aligned}$$

[8]The "*stem*" of a question is a bit that appears before parts **(i)**, **(ii)** etc. Any results given (or proved) in the stem can be used in any or all of the following parts.

Using the initial conditions gives $k = \frac{4}{3}$ and so:

$$\begin{aligned}
\tfrac{4}{3}y^{\frac{3}{2}} &= \tfrac{1}{2}x^2 + \tfrac{4}{3} \\
y^{\frac{3}{2}} &= \tfrac{3}{8}x^2 + 1 \\
y &= \left(\tfrac{3}{8}x^2 + 1\right)^{\frac{2}{3}} \qquad \text{as required}
\end{aligned}$$

(ii) Here there seems to be a similarity to the stem result, but it is less obvious. Perhaps it would help to have a $2y$ next to the $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Multiplying by -2 gives:

$$\begin{aligned}
-2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2y^2 &= 0 \\
-2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2y^2 \qquad (\dagger)
\end{aligned}$$

The left hand side now looks like the stem result, with $n = -2$. This gives:

$$\begin{aligned}
z &= y^{-2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \\
\frac{\mathrm{d}z}{\mathrm{d}x} &= y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}\left(-2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)
\end{aligned}$$

Multiplying $(\dagger)$ by $y^{-3}\dfrac{\mathrm{d}y}{\mathrm{d}x}$ gives:

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}x} &= 2y^2 \times y^{-3}\,\frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{\mathrm{d}z}{\mathrm{d}x} &= 2y^{-1}\,\frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{\mathrm{d}z}{\mathrm{d}x} &= 2z^{\frac{1}{2}}
\end{aligned}$$

Integration (as before) gives $2z^{\frac{1}{2}} = 2x + c$ i.e.:

$$2y^{-1}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + c$$

and the initial conditions give us $c = 0$.

Separating variables again gives:

$$\begin{aligned}
\int \frac{1}{y}\,\mathrm{d}y &= \int x\,\mathrm{d}x \\
\ln y &= \tfrac{1}{2}x^2 + k
\end{aligned}$$

The initial conditions $(x = 0,\ y = 1)$ gives $k = 0$ and so $y = \mathrm{e}^{\left(\frac{1}{2}x^2\right)}$.

5 **2006 S3 Q4**

Setting $y = x$ into the given identity gives:

$$f(x) + f(x) \equiv f(x + x) \implies 2f(x) \equiv f(2x)$$

and then setting $x = 0$ into $2f(x) \equiv f(2x)$ gives:

$$2f(0) = f(0) \implies f(0) = 0\,.$$

The "equivalent to" sign $\equiv$ can be used if a relationship is true for any value(s) - e.g. we can write $2(x + 3) \equiv 2x + 6$ as both sides are equal no matter what value x takes. An equation sign is used when the relationship is true for only some values e.g. $x^2 = 2x$.

The Maclaurin series gives us:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

We have already shown that $f(0) = 0$. Differentiating both sides of $2f(x) \equiv f(2x)$ gives:

$$2f'(x) \equiv 2f'(2x) \quad \text{using the chain rule}$$

Substituting $x = 0$ into this gives $2f'(0) = 2f'(0)$ and so $f'(0)$ can be any constant, k.

Differentiating $2f'(x) \equiv 2f'(2x)$ gives $2f''(x) \equiv 4f''(2x)$ and so $f''(0) = 0$. Similarly all higher derivatives evaluated at $x = 0$ will be equal to 0. Hence we have:

$$f(x) = kx$$

(i) Starting with $g(x)g(y) \equiv g(x + y)$ we have:

$$\begin{aligned} g(x)g(y) &\equiv g(x+y) \\ \implies \ln\big(g(x)\big) + \ln\big(g(y)\big) &= \ln\big(g(x+y)\big) \\ G(x) + G(y) &= G(x+y) \end{aligned}$$

Therefore we have $G(x) = kx$ and so $\ln\big(g(x)\big) = kx \implies g(x) = e^{kx}$.

(ii) Let $x = e^u$ and $y = e^v$. Then we have:

$$\begin{aligned} h(x) + h(y) &= h(xy) \\ \implies h\left(e^u\right) + h\left(e^v\right) &= h\left(e^{(u+v)}\right) \\ \implies H(u) + H(v) &= H(u+v) \end{aligned}$$

Therefore we have $H(u) = ku$ and so $h(e^u) = ku$. Since $x = e^u$ we have $h(x) = k \ln x$.

(iii) In this part we are not told which function to use. However looking at the given $z = \dfrac{x + y}{1 - xy}$ this looks very similar to $\tan(A + B)$.

Let $\mathrm{t}(\tan u) = \mathrm{T}(u)$[9] and let $x = \tan u$ and $y = \tan v$. We can restrict u and v to lie in the range $-\frac{\pi}{2} < u, v < \frac{\pi}{2}$ so that the relationship between x and u (and y and v) is well-defined (i.e. there is only one value of u for each value of x etc.). Then:

$$\begin{aligned}
\mathrm{t}(x) + \mathrm{t}(y) &= \mathrm{t}(z) \\
\mathrm{t}(x) + \mathrm{t}(y) &= \mathrm{t}\left(\frac{x+y}{1-xy}\right) \\
\mathrm{t}(\tan u) + \mathrm{t}(\tan v) &= \mathrm{t}\left(\frac{\tan u + \tan v}{1 - \tan u \tan v}\right) \\
\mathrm{t}(\tan u) + \mathrm{t}(\tan v) &= \mathrm{t}\big(\tan(u+v)\big) \\
\mathrm{T}(u) + \mathrm{T}(v) &= \mathrm{T}(u+v)
\end{aligned}$$

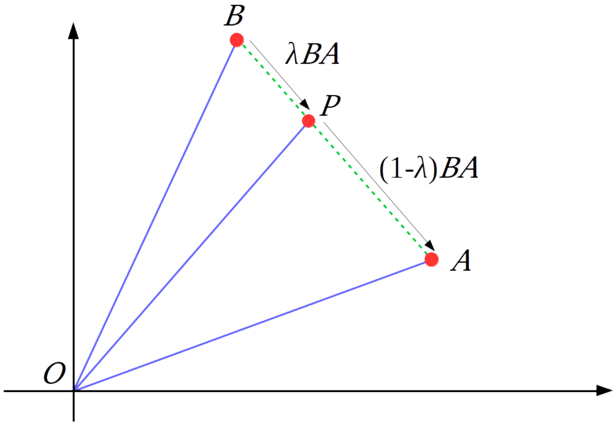
Therefore we have $\mathrm{T}(u) = ku$ and so $\mathrm{t}(x) = k \arctan x$.

Note that the restriction that $-\frac{\pi}{2} < u < \frac{\pi}{2}$ means that $u = \arctan x$. For more on this see question **9**.

[9] The first thing I tried was $\mathrm{t}(x) = \mathrm{T}(\tan x)$ - but this didn't help!

6 2008 S2 Q8

Start by drawing a diagram:



Then using $\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$ gives: $\mathbf{p} = \mathbf{b} + \lambda(\mathbf{a} - \mathbf{b}) = \lambda\mathbf{a} + (\mathbf{1} - \lambda)\mathbf{b}$.

If OP bisects $\angle AOB$ then the angle between OB and OP is equal the angle between OP and angle OA. Using $\mathbf{p} \cdot \mathbf{q} = |\mathbf{p}||\mathbf{q}| \cos\theta$ and equating for $\cos\theta$ gives:

$$\begin{aligned}
\frac{\mathbf{b} \cdot \mathbf{p}}{|\mathbf{b}||\mathbf{p}|} &= \frac{\mathbf{p} \cdot \mathbf{a}}{|\mathbf{p}||\mathbf{a}|} \\
a(\mathbf{b} \cdot \mathbf{p}) &= b(\mathbf{p} \cdot \mathbf{a}) \\
a\big(\mathbf{b} \cdot (\lambda\mathbf{a} + (1-\lambda)\mathbf{b})\big) &= b\big((\lambda\mathbf{a} + (1-\lambda)\mathbf{b}) \cdot \mathbf{a}\big) \\
a(\lambda\mathbf{b} \cdot \mathbf{a} + (1-\lambda)\mathbf{b} \cdot \mathbf{b}) &= b(\lambda\mathbf{a} \cdot \mathbf{a} + (1-\lambda)\mathbf{b} \cdot \mathbf{a}) \\
\mathbf{b} \cdot \mathbf{a}\,(\lambda a + b(\lambda - 1)) &= ab^2(\lambda - 1) + \lambda ba^2 \\
\mathbf{b} \cdot \mathbf{a}\,(\lambda a + b(\lambda - 1)) &= ab\,(b(\lambda - 1) + a\lambda) \\
(\mathbf{b} \cdot \mathbf{a} - ab)\,(\lambda a + b(\lambda - 1)) &= 0
\end{aligned}$$

So either we have $\mathbf{a} \cdot \mathbf{b} = ab$, which is only true if the angle between them is 0 or π, or if $\lambda a + b(\lambda - 1) = 0$.

Since we are told that A, B and O are not collinear then $\mathbf{a} \cdot \mathbf{b} \neq ab$. Hence we must have $\lambda a + b(\lambda - 1) = 0$ and so $\lambda = \dfrac{b}{a+b}$.

When I first tried this question I tried to use the Cosine rule. It got messy and I went wrong somewhere!

We have already shown that $\overrightarrow{OP} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$. By using $\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$ gives $\overrightarrow{OQ} = \lambda\mathbf{b} + (1-\lambda)\mathbf{a}$[10].

[10] You can use different ways of finding $\overrightarrow{OQ}$, such as $\overrightarrow{OQ} = \overrightarrow{OB} + \overrightarrow{BQ}$. They should all result in the same expression for $\overrightarrow{OQ}$.

We then have:

$$\begin{aligned}
OQ^2 - OP^2 &= (\lambda\mathbf{b} + (1-\lambda)\mathbf{a})^2 - (\lambda\mathbf{a} + (1-\lambda)\mathbf{b})^2 \\
&= \lambda^2 b^2 + (1-\lambda)^2 a^2 + \cancel{2\lambda(1-\lambda)\mathbf{b}\cdot\mathbf{a}} - \lambda^2 a^2 - (1-\lambda)^2 b^2 - \cancel{2\lambda(1-\lambda)\mathbf{b}\cdot\mathbf{a}} \\
&= \cancel{\lambda^2 b^2} + a^2 - 2\lambda a^2 + \cancel{\lambda^2 a^2} - \cancel{\lambda^2 a^2} - b^2 + 2\lambda b^2 - \cancel{\lambda^2 b^2} \\
&= a^2 - b^2 + 2\lambda(b^2 - a^2) \\
&= (a^2 - b^2)(1 - 2\lambda) \\
&= (a^2 - b^2)\left(1 - \frac{2b}{a+b}\right) \\
&= (a^2 - b^2)\left(\frac{a+b-2b}{a+b}\right) \\
&= (a+b)(a-b) \times \left(\frac{a-b}{a+b}\right) \\
&= (a-b)^2
\end{aligned}$$

7 2009 S3 Q4

(i) The Laplace transform of $\mathrm{e}^{-bt}\mathrm{f}(t)$ is given by:

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-st} \times \mathrm{e}^{-bt}\mathrm{f}(t)\,\mathrm{d}t &= \int_0^\infty \mathrm{e}^{-(s+b)t}\mathrm{f}(t)\,\mathrm{d}t \\ &= \mathrm{F}(s+b)\end{aligned}$$

(ii) For this part, use the substitution $u = at$

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-st}\mathrm{f}(at)\,\mathrm{d}t &= \int_0^\infty \mathrm{e}^{-s\times\frac{u}{a}}\mathrm{f}(u) \times \tfrac{1}{a}\,\mathrm{d}u \\ &= \frac{1}{a}\int_0^\infty \mathrm{e}^{-\frac{s}{a}u}\mathrm{f}(u)\,\mathrm{d}u \\ &= \frac{1}{a}\mathrm{F}\left(\frac{s}{a}\right)\end{aligned}$$

(iii)

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-st}\mathrm{f}'(t)\,\mathrm{d}t &= \left[\mathrm{e}^{-st}\mathrm{f}(t)\right]_0^\infty + s\int_0^\infty \mathrm{e}^{-st}\mathrm{f}(t)\,\mathrm{d}t \\ &= -\mathrm{f}(0) + s\mathrm{F}(s)\end{aligned}$$

(iv) Let $I = \displaystyle\int_0^\infty \mathrm{e}^{-st}\sin t\,\mathrm{d}t$. Then:

$$\begin{aligned}I &= \left[-\mathrm{e}^{-st}\cos t\right]_0^\infty - s\int_0^\infty \mathrm{e}^{-st}\cos t\,\mathrm{d}t \\ I &= 1 - s\left(\left[\mathrm{e}^{-st}\sin t\right]_0^\infty + s\int_0^\infty \mathrm{e}^{-st}\sin t\,\mathrm{d}t\right) \\ I &= 1 - s(0 + sI) \\ I &= 1 - s^2 I \\ (1+s^2)I &= 1 \\ I &= \frac{1}{1+s^2}\end{aligned}$$

For the last part, we must stick to the previous results shown in the question. We want to find $\int_0^\infty \mathrm{e}^{-st} \times \mathrm{e}^{-pt}\cos qt\,\mathrm{d}t$. Start with the result from part **(iv)**, i.e. the Laplace transform of $\mathrm{f}(t) = \sin t$ is $\dfrac{1}{s^2+1}$.

- Part **(iii)** "$\mathrm{f}'(t) \to s\mathrm{F}(s) - \mathrm{f}(0)$" gives us that the Laplace transform of $\mathrm{f}'(t) = \cos t$ is
$$s\mathrm{F}(s) - \mathrm{f}(0) = \frac{s}{s^2+1} - 0$$
- Part **(ii)** "$\mathrm{f}(at) \to a^{-1}\mathrm{F}\left(\frac{s}{a}\right)$" gives us that the Laplace transform of $\cos(qt)$ is
$$\frac{1}{q} \times \frac{\frac{s}{q}}{\left(\frac{s}{q}\right)^2 + 1} = \frac{s}{s^2+q^2}$$

- Part **(i)** "$\mathrm{e}^{-bt}\mathrm{f}(t) \to \mathrm{F}(s+b)$" gives us that the Laplace transform of $\mathrm{e}^{-pt}\cos(qt)$ is

$$\frac{(s+p)}{(s+p)^2+q^2}$$

Laplace transforms are useful tools in solving differential equations. You can read more about them in this Wikipedia article.

8 2009 S3 Q6

$$
\begin{aligned}
\mathrm{e}^{\mathrm{i}\beta}-\mathrm{e}^{\mathrm{i}\alpha} &= (\cos\beta-\cos\alpha)+\mathrm{i}(\sin\beta-\sin\alpha)\\
\implies |\mathrm{e}^{\mathrm{i}\beta}-\mathrm{e}^{\mathrm{i}\alpha}|^2 &= (\cos\beta-\cos\alpha)^2+(\sin\beta-\sin\alpha)^2\\
&= \cos^2\beta+\cos^2\alpha-2\cos\beta\cos\alpha+\sin^2\beta+\sin^2\alpha-2\sin\beta\sin\alpha\\
&= 2-2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)\\
&= 2(1-\cos(\beta-\alpha))\\
&= 4\sin^2\tfrac12(\beta-\alpha)\\
\implies |\mathrm{e}^{\mathrm{i}\beta}-\mathrm{e}^{\mathrm{i}\alpha}| &= 2\sin\tfrac12(\beta-\alpha)
\end{aligned}
$$

The penultimate step here uses $\cos 2A = 1-2\sin^2 A$. Note that since we want a positive value for $|\mathrm{e}^{\mathrm{i}\beta}-\mathrm{e}^{\mathrm{i}\alpha}|$ we take the positive square root. Since $0<\alpha<\beta<2\pi$ we have $0<\frac12(\beta-\alpha)<\pi$ and so $\sin\frac12(\beta-\alpha)$ is positive.

The first time I did this I ended up with $|\mathrm{e}^{\mathrm{i}\beta}-\mathrm{e}^{\mathrm{i}\alpha}|^2 = 4\sin^2\frac12(\alpha-\beta)$ which means that I would have had to take the negative square root $|\mathrm{e}^{\mathrm{i}\beta}-\mathrm{e}^{\mathrm{i}\alpha}| = -2\sin\frac12(\alpha-\beta)$ to end up with a positive value for the modulus.

For the next part, start with the LHS.

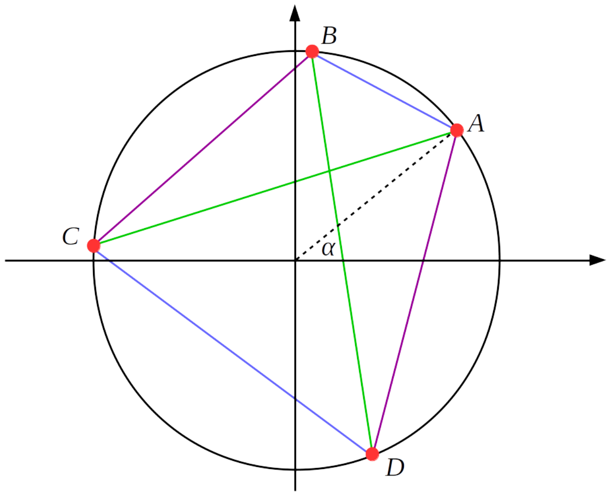
$$
\begin{aligned}
&|\mathrm{e}^{\mathrm{i}\alpha}-\mathrm{e}^{\mathrm{i}\beta}||\mathrm{e}^{\mathrm{i}\gamma}-\mathrm{e}^{\mathrm{i}\delta}|+|\mathrm{e}^{\mathrm{i}\beta}-\mathrm{e}^{\mathrm{i}\gamma}||\mathrm{e}^{\mathrm{i}\alpha}-\mathrm{e}^{\mathrm{i}\delta}|\\
=&|\mathrm{e}^{\mathrm{i}\beta}-\mathrm{e}^{\mathrm{i}\alpha}||\mathrm{e}^{\mathrm{i}\delta}-\mathrm{e}^{\mathrm{i}\gamma}|+|\mathrm{e}^{\mathrm{i}\gamma}-\mathrm{e}^{\mathrm{i}\beta}||\mathrm{e}^{\mathrm{i}\delta}-\mathrm{e}^{\mathrm{i}\alpha}|^{11}\\
=&2\sin\tfrac12(\beta-\alpha)\times 2\sin\tfrac12(\delta-\gamma)+2\sin\tfrac12(\gamma-\beta)\times 2\sin\tfrac12(\delta-\alpha)\\
=&4\Big[\left(\sin\tfrac12\beta\cos\tfrac12\alpha-\sin\tfrac12\alpha\cos\tfrac12\beta\right)\left(\sin\tfrac12\delta\cos\tfrac12\gamma-\sin\tfrac12\gamma\cos\tfrac12\delta\right)\\
&\qquad+\left(\sin\tfrac12\gamma\cos\tfrac12\beta-\sin\tfrac12\beta\cos\tfrac12\gamma\right)\left(\sin\tfrac12\delta\cos\tfrac12\alpha-\sin\tfrac12\alpha\cos\tfrac12\delta\right)\Big]\\
=&4\Big[\cancel{\sin\tfrac12\beta\cos\tfrac12\alpha\sin\tfrac12\delta\cos\tfrac12\gamma}-\sin\tfrac12\alpha\cos\tfrac12\beta\sin\tfrac12\delta\cos\tfrac12\gamma\\
&\qquad-\sin\tfrac12\beta\cos\tfrac12\alpha\sin\tfrac12\gamma\cos\tfrac12\delta+\cancel{\sin\tfrac12\alpha\cos\tfrac12\beta\sin\tfrac12\gamma\cos\tfrac12\delta}\\
&\qquad\quad+\sin\tfrac12\gamma\cos\tfrac12\beta\sin\tfrac12\delta\cos\tfrac12\alpha-\cancel{\sin\tfrac12\beta\cos\tfrac12\gamma\sin\tfrac12\delta\cos\tfrac12\alpha}\\
&\qquad\qquad-\cancel{\sin\tfrac12\gamma\cos\tfrac12\beta\sin\tfrac12\alpha\cos\tfrac12\delta}+\sin\tfrac12\beta\cos\tfrac12\gamma\sin\tfrac12\alpha\cos\tfrac12\delta\Big]\\
=&4\Big[\sin\tfrac12\gamma\cos\tfrac12\beta\sin\tfrac12\delta\cos\tfrac12\alpha-\sin\tfrac12\beta\cos\tfrac12\alpha\sin\tfrac12\gamma\cos\tfrac12\delta\\
&\qquad+\sin\tfrac12\beta\cos\tfrac12\gamma\sin\tfrac12\alpha\cos\tfrac12\delta-\sin\tfrac12\alpha\cos\tfrac12\beta\sin\tfrac12\delta\cos\tfrac12\gamma\Big]\\
=&4\Big[\sin\tfrac12\gamma\cos\tfrac12\alpha\left(\sin\tfrac12\delta\cos\tfrac12\beta-\sin\tfrac12\beta\cos\tfrac12\delta\right)\\
&\qquad-\sin\tfrac12\alpha\cos\tfrac12\gamma\left(\sin\tfrac12\delta\cos\tfrac12\beta-\sin\tfrac12\beta\cos\tfrac12\delta\right)\Big]\\
=&4\left(\sin\tfrac12\gamma\cos\tfrac12\alpha-\sin\tfrac12\alpha\cos\tfrac12\gamma\right)\left(\sin\tfrac12\delta\cos\tfrac12\beta-\sin\tfrac12\beta\cos\tfrac12\delta\right)\\
=&2\sin\tfrac12(\gamma-\alpha)\times 2\sin\tfrac12(\delta-\beta)\\
=&|\mathrm{e}^{\mathrm{i}\gamma}-\mathrm{e}^{\mathrm{i}\alpha}||\mathrm{e}^{\mathrm{i}\delta}-\mathrm{e}^{\mathrm{i}\beta}|\\
=&|\mathrm{e}^{\mathrm{i}\alpha}-\mathrm{e}^{\mathrm{i}\gamma}||\mathrm{e}^{\mathrm{i}\beta}-\mathrm{e}^{\mathrm{i}\delta}|
\end{aligned}
$$

[11] The first step here was to rewrite the different components so that we could use the first result used (which needed $\alpha<\beta$). Note that $|-a|=|a|$.

For the very last part note that a general point on the unit circle can be written as $\mathrm{e}^{\mathrm{i}\alpha}$ where α is the angle anticlockwise from the positive real axis.

The expression $|\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{\mathrm{i}\beta}|$ measures the distance between two points on the unit circle, one with angle α and one with angle β.

The diagram below shows a unit circle with 4 points A, B, C, D which are the points represented by complex numbers $\mathrm{e}^{\mathrm{i}\alpha}, \mathrm{e}^{\mathrm{i}\beta}, \mathrm{e}^{\mathrm{i}\gamma}, \mathrm{e}^{\mathrm{i}\delta}$. Then length $AB = |\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{\mathrm{i}\beta}|$ etc.



The result that we have shown is:

$$|\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{\mathrm{i}\beta}||\mathrm{e}^{\mathrm{i}\gamma} - \mathrm{e}^{\mathrm{i}\delta}| + |\mathrm{e}^{\mathrm{i}\beta} - \mathrm{e}^{\mathrm{i}\gamma}||\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{\mathrm{i}\delta}| = |\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{\mathrm{i}\gamma}||\mathrm{e}^{\mathrm{i}\beta} - \mathrm{e}^{\mathrm{i}\delta}|$$

which means that:

$$AB \times CD + BC \times AD = AC \times BD$$

This means that the products of the lengths of the diagonals is equal to the sum of the products of the two pairs of opposite sides.

We have shown this to be true in the unit circle, and since all circles are similar this is true for any cyclic quadrilateral. Alternatively you can write the vertices on a general circle with centre the origin as $r\mathrm{e}^{\mathrm{i}\theta}$, and then all the r's would cancel out.

This theorem is also known as Ptolemy's Theorem. You were not expected to be familiar with this theorem!

9 2012 S3 Q8

(i) Nothing "high tech" needed here. The first five Fibonacci numbers are $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$ and $F_5 = 5$.

Then we have:

$$F_0F_3 - F_1F_2 = 0 \times 2 - 1 \times 1 = -1$$
$$F_2F_5 - F_3F_4 = 1 \times 5 - 2 \times 3 = -1$$

which are the same, so $F_0F_3 - F_1F_2 = F_2F_5 - F_3F_4$.

(ii) To start with, it looks like one of the values is -1. Consider the case "between" the two in part **(i)** and we have $F_1F_4 - F_2F_3 = 1 \times 3 - 1 \times 2 = 1$. We can then conjecture that:

$$F_nF_{n+3} - F_{n+1}F_{n+2} = (-1)^{n+1}$$

i.e. it is equal to 1 when n is odd and -1 when n is even. We now need to prove this.

If we can show that $F_{n+2}F_{n+5} - F_{n+3}F_{n+4} = F_nF_{n+3} - F_{n+1}F_{n+2}$, then we can use an induction argument to prove our conjecture.

$$\begin{aligned}
F_{n+2}F_{n+5} - F_{n+3}F_{n+4} &= (F_n + F_{n+1})(F_{n+3} + F_{n+4}) - (F_{n+1} + F_{n+2})(F_{n+2} + F_{n+3}) \\
&= F_nF_{n+3} + \cancel{F_{n+1}F_{n+3}} + F_nF_{n+4} + F_{n+1}F_{n+4} \\
&\qquad - F_{n+1}F_{n+2} - F_{n+2}F_{n+2} - \cancel{F_{n+1}F_{n+3}} - F_{n+2}F_{n+3} \\
&= F_nF_{n+3} - F_{n+1}F_{n+2} + F_{n+4}(F_n + F_{n+1}) - F_{n+2}(F_{n+2} + F_{n+3}) \\
&= F_nF_{n+3} - F_{n+1}F_{n+2} + \cancel{F_{n+4}F_{n+2}} - \cancel{F_{n+2}F_{n+4}} \\
&= F_nF_{n+3} - F_{n+1}F_{n+2}
\end{aligned}$$

What you cannot see here is the crossings out and wrong turns I made before getting this to work. Please don't expect to get the answer out first time!

Hence we have $F_nF_{n+3} - F_{n+1}F_{n+2} = F_{n+2}F_{n+5} - F_{n+3}F_{n+4}$. This means that we have $F_0F_3 - F_1F_2 = F_2F_5 - F_3F_4 = F_4F_7 - F_5F_6 = \ldots = -1$ and $F_1F_4 - F_2F_3 = F_3F_6 - F_4F_5 = F_5F_8 - F_6F_7 = \ldots = 1$ and so:

$$F_nF_{n+3} - F_{n+1}F_{n+2} = \begin{cases} -1 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

(iii) Using the identity for $\tan(A+B)$ we have:

$$\begin{aligned}
\tan\left(\arctan\left(\tfrac{1}{F_{2r+1}}\right)+\arctan\left(\tfrac{1}{F_{2r+2}}\right)\right) &= \frac{\tan\left[\arctan\left(\tfrac{1}{F_{2r+1}}\right)\right]+\tan\left[\arctan\left(\tfrac{1}{F_{2r+2}}\right)\right]}{1-\tan\left[\arctan\left(\tfrac{1}{F_{2r+1}}\right)\right]\tan\left[\arctan\left(\tfrac{1}{F_{2r+2}}\right)\right]} \\
&= \frac{\frac{1}{F_{2r+1}}+\frac{1}{F_{2r+2}}}{1-\frac{1}{F_{2r+1}}\frac{1}{F_{2r+2}}} \\
&= \frac{F_{2r+2}+F_{2r+1}}{F_{2r+1}F_{2r+2}-1}
\end{aligned}$$

The numerator of this is equal to F_{2r+3}. For the denominator, we know that $F_{2r}F_{2r+3}-F_{2r+1}F_{2r+2}=-1$ (since $2r$ is even) and so $F_{2r+1}F_{2r+2}-1=F_{2r}F_{2r+3}$. We now have:

$$\begin{aligned}
\tan\left(\arctan\left(\tfrac{1}{F_{2r+1}}\right)+\arctan\left(\tfrac{1}{F_{2r+2}}\right)\right) &= \frac{F_{2r+3}}{F_{2r}F_{2r+3}} \\
&= \frac{1}{F_{2r}} \\
&= \tan\left(\arctan\left(\tfrac{1}{F_{2r}}\right)\right)
\end{aligned}$$

A little care is needed now, since $\tan A=\tan B$ does not necessarily imply that $A=B$! Each of the "arctans" has a value between 0 and $\frac{\pi}{2}$, and so $\arctan\left(\frac{1}{F_{2r+1}}\right)+\arctan\left(\frac{1}{F_{2r+2}}\right)$ has a value between 0 and π. Between these values $\tan x$ is a "one-to-one" function, and so in this case $\tan A=\tan B$ **does** imply that $A=B$ and hence we can conclude:

$$\arctan\left(\tfrac{1}{F_{2r+1}}\right)+\arctan\left(\tfrac{1}{F_{2r+2}}\right)=\arctan\left(\tfrac{1}{F_{2r}}\right)$$

For the final part, we have:

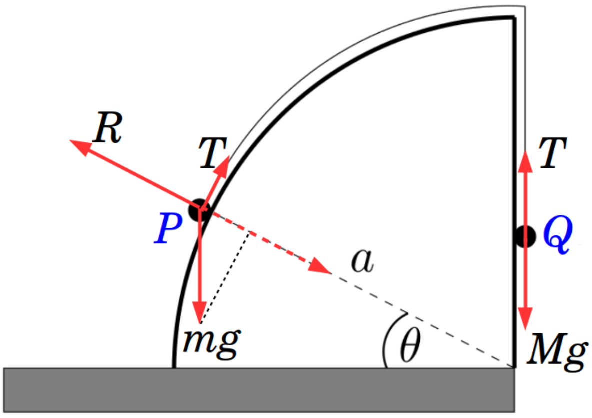
$$\begin{aligned}
\sum_{r=1}^{\infty}\arctan\left(\frac{1}{F_{2r+1}}\right) &= \sum_{r=1}^{\infty}\left[\arctan\left(\frac{1}{F_{2r}}\right)-\arctan\left(\frac{1}{F_{2r+2}}\right)\right] \\
&= \arctan\left(\frac{1}{F_2}\right)+\arctan\left(\frac{1}{F_4}\right)+\arctan\left(\frac{1}{F_6}\right)+\dots \\
&\qquad\qquad -\arctan\left(\frac{1}{F_4}\right)-\arctan\left(\frac{1}{F_6}\right)-\dots \\
&= \arctan\left(\frac{1}{F_2}\right) \\
&= \arctan(1) \\
&= \tfrac{1}{4}\pi
\end{aligned}$$

It is true that $\tan(\arctan x)=x$, but not necessarily true that $\arctan(\tan x)=x$. Try evaluating these for some values of x (calculators allowed, but start without a calculator using $x=1$ and $x=\sqrt{3}$ for the first expression and $x=\frac{1}{4}\pi$ and $x=\frac{3}{4}\pi$ for the second). Try sketching some graphs to see what is happening. What about $\sin(\arcsin x)$ and $\arcsin(\sin x)$?

10 2010 S3 Q9

First thing to note is that the string is "light" (which means we do not have to worry about the effect of gravity on the string) and "in-extensible" (so the two particles move with the same acceleration and the tension is the same throughout the string). The block is "smooth" so we do not need to worry about friction. Since $m < M$ when the particles are released Q moves downwards.

The next step is to draw a clear diagram showing the forces acting on the particles. Be very careful to ensure that "m" and "M" are easily distinguishable![12]



Using "$F = ma$" radially for P gives us:

$$\frac{mv^2}{a} = mg\sin\theta - R \qquad (*)$$

When P has moved through an angle of θ, it has moved an arc length of $a\theta$ round the quadrant. Hence Q will have moved down by $a\theta$. Conservation of energy gives us:

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}Mv^2 + mg(a\sin\theta) - Mg(a\theta) = 0$$

$$\text{i.e.} \quad \tfrac{1}{2}\left(\frac{v^2}{a}\right)(m+M) = g(M\theta - m\sin\theta)$$

The question wants us to find an expression for R in terms of m, M, θ and g, so we want to eliminate v^2 and a. Equation $(*)$ gives us $\frac{v^2}{a} = g\sin\theta - \frac{1}{m}R$, and substituting into the conservation of energy equation gives:

$$\tfrac{1}{2}\left(g\sin\theta - \frac{1}{m}R\right)(m+M) = g(M\theta - m\sin\theta)$$

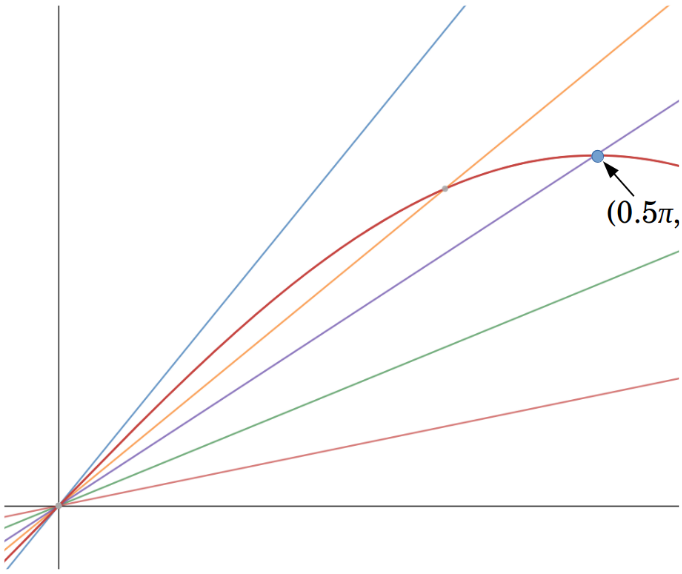
[12]Some people use the substitution $m = m_1$, $M = m_2$ to help ensure that they do not get confused in the middle of their working. This is fine as long as you make it clear that this is what you are doing!

Rearranging gives:

$$\begin{aligned}\left(g\sin\theta - \frac{1}{m}R\right) &= \frac{2g(M\theta - m\sin\theta)}{(m+M)}\\ \implies R = mg\sin\theta - \frac{2mg(M\theta - m\sin\theta)}{(m+M)}\\ &= \frac{m^2g\sin\theta + mMg\sin\theta - 2mMg\theta + 2m^2g\sin\theta}{(m+M)}\\ &= \frac{mg\Big[(3m+M)\sin\theta - 2M\theta\Big]}{m+M}\end{aligned}$$

For P to remain in contact with the block, we need $R \geqslant 0$ for all $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Hence we need $(3m+M)\sin\theta - 2M\theta \geqslant 0$ in this range. Equivalently we can write this as $\sin\theta \geqslant \frac{2M}{3m+M}\theta$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Let $\lambda = \frac{2M}{3m+M}$.

Below is a sketch of $y = \sin\theta$ and $y = \lambda\theta$ for some values of λ.



From this we can see that as long as $\sin\theta \geqslant \lambda\theta$ when $\theta = \frac{1}{2}\pi$ then $\sin\theta \geqslant \lambda\theta$ throughout the range $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Hence we need $1 \geqslant \lambda \times \frac{1}{2}\pi$ and so $\lambda \leqslant \frac{2}{\pi}$.

We therefore need:

$$\begin{aligned}\frac{2M}{3m+M} &\leqslant \frac{2}{\pi}\\ 2\pi M &\leqslant 6m + 2M\\ 2M(\pi - 1) &\leqslant 6m\\ \frac{2(\pi-1)}{6} &\leqslant \frac{m}{M}\end{aligned}$$

and hence we have $\dfrac{m}{M} \geqslant \dfrac{\pi - 1}{3}$ as required.

See STEP solutions from MEI (http://mei.org.uk/step-aea-solutions) for a slightly different solution to this question (which doesn't use conservation of momentum).

11 2008 S3 Q13

There are $2n$ string ends altogether initially. Start by thinking what happens with the first two string ends selected. If I pick one string end then the probability that I will pick the other end of that string and hence make a loop/ring is $\dfrac{1}{2n-1}$ and the probability that I will pick an end from a different string and hence make a longer string is $\dfrac{2n-2}{2n-1}$.

Then, on the next step of the process I now have $2n-2$ ends to consider (as regardless of whether I made a ring or a longer string last time I only have $2n-2$ "free ends"). If I select one end at random and consider the $2n-3$ other ends the probability I make a ring is $\dfrac{1}{2n-3}$ and the probability that I make a longer string is $\dfrac{2n-4}{2n-3}$. The process will be repeated exactly n times.

At each stage, the number of rings made is independent of the number of rings made previously (this is important for the variance as it means that we can use $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$).

On the first step the expected number of rings made is:

$$0 \times \frac{2n-2}{2n-1} + 1 \times \frac{1}{2n-1} = \frac{1}{2n-1}\,.$$

Similarly, on the second step of the process the expected number of rings made is $\dfrac{1}{2n-3}$, on the third step the expected number of rings made is $\dfrac{1}{2n-5}$ etc. Therefore the expected number of rings made by the end of the process is:

$$\frac{1}{2n-1} + \frac{1}{2n-3} + \frac{1}{2n-5} + \cdots + \frac{1}{3} + 1$$

The variance of the number of rings made on the first step of the process is:

$$0^2 \times \frac{2n-2}{2n-1} + 1^2 \times \frac{1}{2n-1} - \left(\frac{1}{2n-1}\right)^2$$

The variance of the number of rings made by the end of the process is:

$$\left[\frac{1}{2n-1} - \frac{1}{(2n-1)^2}\right] + \left[\frac{1}{2n-3} - \frac{1}{(2n-3)^2}\right] + \left[\frac{1}{2n-5} - \frac{1}{(2n-5)^2}\right] + \cdots + \left[\frac{1}{1} - \frac{1}{1^2}\right]$$
$$= \frac{2(n-1)}{(2n-1)^2} + \frac{2(n-2)}{(2n-3)^2} + \frac{2(n-3)}{(2n-5)^2} + \cdots + \frac{2}{3^2} + 0$$

When $n = 40000$ the expectation is:

$$\frac{1}{79999} + \frac{1}{79997} + \cdots + \frac{1}{5} + \frac{1}{3} + \frac{1}{1}$$
$$= 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{79997} + \frac{1}{79999}$$

We are told that $1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n$. We can write the expectation as:

$$\begin{aligned}
&1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{79997} + \frac{1}{79999} \\
&= \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{79998} + \frac{1}{79999} + \frac{1}{80000}\right] - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{79998} + \frac{1}{80000}\right] \\
&= \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{79998} + \frac{1}{79999} + \frac{1}{80000}\right] - \tfrac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{3} + + \cdots + \frac{1}{39999} + \frac{1}{40000}\right] \\
&\approx \ln(80000) - \tfrac{1}{2}\ln(40000)
\end{aligned}$$

The other bit of information we were given is that $\ln 20 \approx 3$, so now try to use this in the approximation for the expectation.

$$\begin{aligned}
\ln(80000) - \tfrac{1}{2}\ln(40000) &= \ln(80000) - \ln\sqrt{40000} \\
&= \ln(80000) - \ln(200) \\
&= \ln\left(\tfrac{80000}{200}\right) \\
&= \ln 400 \\
&= 2\ln 20 \\
&\approx 6
\end{aligned}$$

Hence the expected number of rings is approximately 6 (which, to me at least, is a surprisingly low number!).

The first time I did this questions I left out the $\frac{1}{80000}$ terms, which did not work out well.