STEP Support Programme

Pure STEP 1 Questions

2012 S1 Q4

1 Preparation

(i) Find the equation of the tangent to the curve $y = \sqrt{x}$ at the point where $x = 4$.

Recall that $\sqrt{x}$ means the positive square root.

(ii) Solve the simultaneous equations $ax + by = c$ and $y = 2ax$, where $b \neq -\frac{1}{2}$.

(iii) Show that the equation of the normal to the curve $y = x^2$ at the point where $x = p$ is $2py + x = 2p^3 + p$.

(iv) Show that $(a - b)(a^2 + ab + b^2) = a^3 - b^3$. Simplify (for $a \neq b$):

$$\frac{a^3 - b^3}{a^2 - b^2}\,.$$

2 The STEP I question

The curve C has equation $xy = \frac{1}{2}$. The tangents to C at the distinct points $P\big(p, \frac{1}{2p}\big)$ and $Q\big(q, \frac{1}{2q}\big)$, where p and q are positive, intersect at T and the normals to C at these points intersect at N. Show that T is the point

$$\left(\frac{2pq}{p+q}\,,\,\frac{1}{p+q}\right).$$

In the case $pq = \frac{1}{2}$, find the coordinates of N. Show (in this case) that T and N lie on the line $y = x$ and are such that the product of their distances from the origin is constant.

Discussion

Note that your answers for the points of intersection should be symmetrical in p and q, i.e. you should get the same result when you switch p and q around. This is because you get the same point of intersection if you swap the points P and Q. This is a good check of your algebra.

Note also that if $p = q$ (which is not allowed in the question) then P and Q are the same point, the two tangents are concurrent (i.e. they are the same line), and every point on the tangent is a possible solution of the equations. This suggests that every term will have a factor of $p - q$ so that the equations are automatically satisfied when $p = q$. This factor can be divided out, though when you do so you should state 'p and q are distinct so $p - q \neq 0$'.

When finding the coordinates of N, you can choose to use $pq = \frac{1}{2}$ at different points in your argument, some points will involve less algebraic manipulation than others. It may be helpful to note that $p^2 + q^2 + pq = (p + q)^2 - pq$.

2012 S1 Q8

3 Preparation

(i) Use the substitution $y = xv$, where v is a function of x, to show that the differential equation:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^3 \qquad (x \neq 0) \qquad (*)$$

becomes:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = x.$$

You get two terms when you use the product rule to differentiate xv.

Find v in terms of x and hence find y in terms of x. Substitute your solution for y into $(*)$ to show that it does indeed satisfy the original equation.

(ii) *Separation of Variables*

A differential equation of the form (or one that can be put into the form):

$$\mathrm{f}(y)\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{g}(x) \qquad (\dagger)$$

can be solved this by 'separating the variables'. What you do is to write it as

$$\mathrm{f}(y)\mathrm{d}y = \mathrm{g}(x)\mathrm{d}x$$

and integrate both sides.

Although this is what you do in practice, it doesn't really make much sense: what does the above equation mean??

To justify the method, we have to ask first what $(\dagger)$ means. The solution to the equation will be a formula for y as a function of x, say $y = \mathrm{h}(x)$. Therefore, we should really write $(\dagger)$ as

$$\mathrm{f}(\mathrm{h}(x))\mathrm{h}'(x) = \mathrm{g}(x).$$

Integrating with respect to x gives

$$\int \mathrm{f}(\mathrm{h}(x))\mathrm{h}'(x)\,\mathrm{d}x = \int \mathrm{g}(x)\,\mathrm{d}x.$$

In Assignment 25 (Integration by Substitution) we saw that the left hand side of this equation is just what you get if you make the substitution $y = \mathrm{h}(x)$ in the integral $\int \mathrm{f}(y)\mathrm{d}y$. We can therefore write the equation as

$$\int \mathrm{f}(y)\mathrm{d}y = \int \mathrm{g}(x)\mathrm{d}x\,,$$

in agreement with what we said in the first place.

Use this technique to solve:

(a) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = y + 1$, giving your answer in the form $y = \cdots$.

Note that $\mathrm{e}^{\ln x + c} = \mathrm{e}^c \times \mathrm{e}^{\ln x} = Ax$ where A is a constant such that $A = \mathrm{e}^c$.

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \tan y\,.$

Recall that $\tan y = \dfrac{\sin y}{\cos y}\,.$

(iii) Find A and B such that $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}.$

Hence evaluate $\displaystyle\int_1^2 \frac{x}{(x+1)(x+2)}\,\mathrm{d}x\,,$ giving your answer in the form $\ln\left(\frac{m}{n}\right)$.

4 The STEP I question

(i) Show that substituting $y = xv$, where v is a function of x, in the differential equation

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 - 2x^2 = 0 \qquad (x \neq 0)$$

leads to the differential equation

$$xv\frac{\mathrm{d}v}{\mathrm{d}x} + 2v^2 - 2 = 0\,.$$

Hence show that the general solution can be written in the form

$$x^2(y^2 - x^2) = C\,,$$

where C is a constant.

(ii) Find the general solution of the differential equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x} + 6x + 5y = 0 \qquad (x \neq 0).$$

Discussion

Note that the first differential equations in y cannot be solved by separating the variables, so the point of making the substitutions is to achieve an equation for which we can separate the variables. Part **(ii)** is very similar to part **(i)**. As in part **(i)**, you should leave your answer to part **(ii)** as an *implicit* equation; don't try to re-write it as $y = \cdots$. However, you should write it in a form which does not involve fractions (make it look nice!).

2010 S1 Q5

5 Preparation

(i) Find an expression for the infinite sum $1 + p + p^2 + \cdots$, where $-1 < p < 1$. Use this to show that $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$.

(ii) By differentiating your expression in part **(i)** with respect to p, find an expression for:

$$1 + 2p + 3p^2 + \cdots .$$

Hence evaluate $1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \cdots$.

(iii) Expand $(1+x)^5$. By integrating this result, show that:

$$x + \tfrac{5}{2}x^2 + \tfrac{10}{3}x^3 + \tfrac{5}{2}x^4 + x^5 + \tfrac{1}{6}x^6 + c \equiv \tfrac{1}{6}(1+x)^6 ,$$

where c is a constant of integration. By choosing a value for x, find c.

6 The STEP I question

By considering the expansion of $(1+x)^n$ where n is a positive integer, or otherwise, show that:

(i) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$;

(ii) $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$;

(iii) $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{1}{n+1}\left(2^{n+1} - 1\right)$;

(iv) $\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n} = n\,(n+1)\,2^{n-2}$.

Note: $\binom{n}{r}$ is a notation for $\dfrac{n!}{r!(n-r)!}$ and the binomial expansion is

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

2014 S1 Q1

7 Preparation

Any even number can be written in the form $2k$ and any odd number can be written in the form $2k+1$, where k is an integer.

(i) By considering $(2a+1)(2b+1)$, prove that the product of any two odd numbers is an odd number.

Note that $2a+1$ and $2b+1$ represent any two odd numbers, even the same odd numbers if $a = b$. A common mistake, when tackling this sort of question, is to use $(2n+1)$ and $(2n-1)$ but then you have only shown that the product of two *consecutive* odd numbers is odd, i.e. you have show a proof that works for 5×7 but not 3×13.

(ii) Prove that the sum of two odd numbers is an even number.

(iii) Simplify $(2k+1)^2 - (2k-1)^2$ and hence prove that any number of the form $8k$ can be written as the difference of two odd squares.

(iv) Show that $(2a+1)^2 - (2b+1)^2 = 4(a-b)(a+b+1)$. Hence prove that the difference between any two odd squares is divisible by 8.

Exactly one of the two brackets $(a-b)$ and $(a+b+1)$ is even. You should explain why this is true. Note that the direction of proof here is opposite to that in part **(iii)**.

(v) State and prove a conjecture about the difference of two even squares.

(vi) Write down the value of $58^2 - 42^2$.

(vii) If $ab = 12$, $a > b$, and a and b are non-negative integers, find the possible values of a and b.

8 The STEP I question

All numbers referred to in this question are non-negative integers.

(i) Express each of the numbers 3, 5, 8, 12 and 16 as the difference of two non-zero squares.

(ii) Prove that any odd number can be written as the difference of two squares.

(iii) Prove that all numbers of the form $4k$, where k is a non-negative integer, can be written as the difference of two squares.

(iv) Prove that no number of the form $4k + 2$, where k is a non-negative integer, can be written as the difference of two squares.

(v) Prove that any number of the form pq, where p and q are prime numbers greater than 2, can be written as the difference of two squares in exactly two distinct ways. Does this result hold if p is a prime greater than 2 and $q = 2$?

(vi) Determine the number of distinct ways in which 675 can be written as the difference of two squares.

Discussion

For part **(iv)**, if $a^2 - b^2$ is your difference of two squares, you can consider the 4 cases depending on whether a and/or b is odd/even. You have already considered two of these cases in question **1**.

2005 S1 Q4

9 Preparation

(i) If $\tan 3\theta = 1$, what are the possible values of θ in the range $0 \leqslant \theta < 2\pi$?

(ii) If $\sin\theta = \frac{1}{2}$ and $\cos\theta < 0$, what are the possible values of θ?

(iii) Show that $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$. By replacing θ by $\frac{1}{2}\pi - \theta$, deduce from this result an expression for $\cos 3\theta$.

(iv) By spotting one solution (which may not be an integer), and using it to reduce the cubic to a quadratic, solve the equations:

(a) $x^3 - 2x^2 - 11x + 12 = 0$
(b) $12x^3 + 11x^2 - 2x - 1 = 0$
(c) $2x^3 - 5x^2 + 1 = 0$

(v) Show that $-1 < 8 - 5\sqrt{3} < 0$.

10 The STEP I question

(i) Given that $\cos\theta = \dfrac{3}{5}$ and that $\dfrac{3\pi}{2} \leqslant \theta \leqslant 2\pi$, show that $\sin 2\theta = -\dfrac{24}{25}$, and evaluate $\cos 3\theta$.

(ii) Prove the identity $\tan 3\theta \equiv \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

Hence evaluate $\tan\theta$, given that $\tan 3\theta = \dfrac{11}{2}$ and that $\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{\pi}{2}$.

2013 S1 Q3

11 Preparation

(i) Point A has position vector $\mathbf{a}$ (i.e. $\overrightarrow{OA} = \mathbf{a}$), and point B has position vector $\mathbf{b}$.

In the following questions, you will find diagrams useful.

(a) Find (in terms of $\mathbf{a}$ and $\mathbf{b}$) the vector $\overrightarrow{AB}$.

(b) Find the position vector of point C which lies $\frac{2}{3}$ of the way along the line segment AB (so that C is closer to B than to A).

(c) Draw a diagram showing the location of the point with position vector $\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$ in the two cases $0 < \lambda < 1$ and $\lambda > 1$.

(ii) The binary[1] operation $\oplus$ is defined by:

$$a \oplus b = ab - a\,.$$

(a) Find $2 \oplus 3$ and $3 \oplus 2$.

(b) Find the conditions for which $a \oplus b$ is equal to $b \oplus a$.

(c) Find $3 \oplus (5 \oplus 1)$ and $(3 \oplus 5) \oplus 1$.

(d) Find the conditions under which $a \oplus (b \oplus c)$ and $(a \oplus b) \oplus c$ are distinct; i.e. $a \oplus (b \oplus c) \neq (a \oplus b) \oplus c$.

[1] A binary operation is one that acts on two objects. Multiplication, subtraction, vector dot product are all examples.

12 The STEP I question

For any two points X and Y, with position vectors $\mathbf{x}$ and $\mathbf{y}$ respectively, $X * Y$ is defined to be the point with position vector $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$, where λ is a fixed number.

(i) If X and Y are distinct, show that $X * Y$ and $Y * X$ are distinct unless λ takes a certain value (which you should state).

(ii) Under what conditions are $(X * Y) * Z$ and $X * (Y * Z)$ distinct?

(iii) Show that, for any points X, Y and Z,

$$(X * Y) * Z = (X * Z) * (Y * Z)$$

and obtain the corresponding result for $X * (Y * Z)$.

(iv) The points P_1, P_2, ... are defined by $P_1 = X * Y$ and, for $n \geqslant 2$, $P_n = P_{n-1} * Y$. Given that X and Y are distinct and that $0 < \lambda < 1$, find the ratio in which P_n divides the line segment XY.

Discussion

This question looks particularly frightening, because it involves not only vectors but a completely new operation defined in the question. However, those who were brave enough to try it in the examination were usually well rewarded.

The sticking point was often the last request in part (iii); here you have to take a shot in the dark (always tricky under examination conditions), but a calm look at the previous result should suggest something (which you would then have to verify).

Part (iv) could be done geometrically or (if you work out P_2 and P_3, say, and make a conjecture) by induction.

2008 S1 Q4

13 Preparation

(i) Solve the equation $2\sin^2 x + 3\cos x = 0$ where $0 \leqslant x < 2\pi$.

(ii) Write $\sin^2 x$ in terms of $\cos 2x$. Hence sketch $\sin^2 x$ for $0 \leqslant x \leqslant 2\pi$.

(iii) If $\mathrm{f}(x) = \mathrm{e}^{\cos 2x}$, find $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$.

(iv) If $\mathrm{f}(x) = \cos(\tan x)$, find $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$.

(v) Show that $\cot\alpha = \tan(\frac{\pi}{2} - \alpha)$.

(vi) Solve the equation $\sec^2 x = 2\tan x$ for $0 \leqslant x < 2\pi$.

14 The STEP I question

A function $\mathrm{f}(x)$ is said to be *convex* in the interval $a < x < b$ if $\mathrm{f}''(x) \geqslant 0$ for all x in this interval.

(i) Sketch on the same axes the graphs of $y = \frac{2}{3}\cos^2 x$ and $y = \sin x$ in the interval $0 \leqslant x \leqslant 2\pi$.

The function $\mathrm{f}(x)$ is defined for $0 < x < 2\pi$ by

$$\mathrm{f}(x) = \mathrm{e}^{\frac{2}{3}\sin x}.$$

Determine the intervals in which $\mathrm{f}(x)$ is convex.

(ii) The function $\mathrm{g}(x)$ is defined for $0 < x < \frac{1}{2}\pi$ by

$$\mathrm{g}(x) = \mathrm{e}^{-k\tan x}.$$

If $k = \sin 2\alpha$ and $0 < \alpha < \frac{1}{4}\pi$, show that $\mathrm{g}(x)$ is convex in the interval $0 < x < \alpha$, and give one other interval in which $\mathrm{g}(x)$ is convex.

Discussion

In the second part, it will probably be easier to leave k as k (i.e. not use $k = \sin 2\alpha$) until you have finished differentiating. Although the second part does not ask you to draw a sketch, you may find a sketch is useful in the later stages of the question.

2013 S1 Q7

15 Preparation

Some of this preparation may not be strictly relevant to the way you decide to do the STEP question.

(i) How many different ways can you find to integrate $\displaystyle\int \frac{x}{1-x^2}\,\mathrm{d}x$?

(ii) Show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{x} \quad (x > 0, \quad x^2 \neq \mathrm{e})$$

which satisfies $y = 2$ when $x = 1$ is $y = \dfrac{2}{1 - 2\ln x}$. Sketch this graph.

(iii) Find y in terms of x, given that

$$\frac{y^2}{3x^3} = -\frac{1}{x^2} + 1,$$

and determine the values of x for which y is real.

16 The STEP I question

(i) Use the substitution $y = ux$, where u is a function of x, to show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$ is

$$y = x\sqrt{4 + 2\ln x} \qquad (x > \mathrm{e}^{-2}).$$

(ii) Use a substitution to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

(iii) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

Discussion

You should explain why the positive square root is taken, and also you should state the domain for which the solutions hold, such as given in part **(i)** where the domain is given as $x > e^{-2}$.

You can look for a different substitution for part **(ii)** if you like, but you don't have to.

You will need a substitution for part **(iii)** as well, and you will find that the substitution $y = ux$ won't give a differential equation in which the variables are separable. There are several ways to proceed, including finding a suitable substitution. A bit of trial and error might be necessary, but do remember that these questions are supposed to take 30 to 45 minutes so you will have time to experiment. It will help to have a good look at the substitution in part **(ii)** to see how it works.