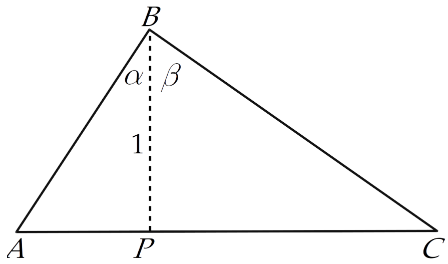
STEP Support Programme

Assignment 10

Warm-up

1 **(i)** In the triangle below, BP is perpendicular to AC.



Show, using the sine rule, that:

$$\frac{\sin(\alpha+\beta)}{\tan\alpha+\tan\beta} = \frac{\sin(90^\circ-\beta)}{\frac{1}{\cos\alpha}}$$

and hence that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha\,.$$

Use this result, together with properties of sin and cos such as $\cos\gamma = \sin(90^\circ - \gamma)$, to obtain expressions for $\sin(\alpha-\beta)$, $\cos(\alpha+\beta)$ and $\cos(\alpha-\beta)$.

(ii) Find the values of $\sin 75^\circ$ and $\sin 15^\circ$.

(iii) By expressing $\cos 3A$ as $\cos(2A + A)$, find an expression for $\cos 3A$ in terms of $\cos A$.

Preparation

2 **(i)** Write down the prime factors of 120. Find the value of:

$$120\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$

(ii) Show that if x is an integer and a is a positive integer, then $x^a\left(1-\dfrac{1}{x}\right)$ is an integer.

(iii) Write 39600 and 52920 as products of prime numbers. Hence find the highest common factor of 39600 and 52920.

(iv) The phrases "if" and "only if" mean very different things. For example, the statement

$$a^2 = b^2 \textbf{ if } a = b$$

is true, but the statement

$$a^2 = b^2 \textbf{ only if } a = b$$

is **not** true (because $a^2 = b^2$ **also** when $a = -b$).

State whether the following are true or false. Express your answers using $\Rightarrow$ ('implies'), $\Leftarrow$ ('is implied by'), $\not\Rightarrow$ ('does not imply) and $\not\Leftarrow$ ('is not implied by') .

For example,

$$a^2 = b^2 \textbf{ if } a = b$$

is True: $a^2 = b^2 \Leftarrow a = b$ and

$$a^2 = b^2 \textbf{ only if } a = b$$

is False: $a^2 = b^2 \not\Rightarrow a = b$.

(a) ab is even **if** both a and b are even

(b) ab is even **only if** both a and b are even

(c) $a = b$ **if** $a^2 = b^2$

(d) $a = b$ **only if** $a^2 = b^2$

(e) a triangle is equilateral **if** it has three equal sides

(f) a triangle is equilateral **only if** it has three equal sides.

It can sometimes be helpful to "rephrase" the statement, so "$a^2 = b^2$ **if** $a = b$" becomes "**if** $a = b$ **then** $a^2 = b^2$" and "$a^2 = b^2$ **only if** $a = b$" becomes "**only if** $a = b$ is it true that $a^2 = b^2$".

(v) Sometimes the implication works both ways. We can use the phrase **if and only if** in these cases. For example:

a triangle is scalene **if and only if** no two sides have equal lengths

Which means that the statements "**if** a triangle is scalene **then** no two sides have equal lengths" and "**if** no two sides have equal lengths **then** a triangle is scalene" are both true.

'If and only if' is sometimes abbreviated as "iff" or denoted by the symbol $\iff$.

State whether the following are true or false:

(a) an even number is prime **iff** it is 2

(b) an odd number is prime **iff** it is 3

(c) $x = 3$ **iff** $x^2 - 9 = 0$

(d) a triangle with sides of lengths a, b and c is right-angled **iff** $a^2 + b^2 = c^2$

(e) a triangle is scalene **iff** no two angles are the same.

For those parts where the statement is false, re-write the statement using "if" or "only if" to make it true.

The STEP question

3 For any positive integer N, the function $\mathrm{f}(N)$ is defined by

$$\mathrm{f}(N) = N\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right)$$

where p_1, p_2, ... , p_k are the only prime numbers that are factors of N.
Thus $\mathrm{f}(80) = 80(1 - \frac{1}{2})(1 - \frac{1}{5})$.

(i) **(a)** Evaluate $\mathrm{f}(12)$ and $\mathrm{f}(180)$.

(b) Show that $\mathrm{f}(N)$ is an integer for all N.

(ii) Prove, or disprove by means of a counterexample, each of the following:

(a) $\mathrm{f}(m)\mathrm{f}(n) = \mathrm{f}(mn)$;

(b) $\mathrm{f}(p)\mathrm{f}(q) = \mathrm{f}(pq)$ if p and q are distinct prime numbers;

(c) $\mathrm{f}(p)\mathrm{f}(q) = \mathrm{f}(pq)$ only if p and q are distinct prime numbers.

(iii) Find a positive integer m and a prime number p such that $\mathrm{f}(p^m) = 146410$.

Discussion

Again, a STEP question where a (possibly) unfamiliar function is presented to you. Don't be frightened of the last part: it is easier than it looks!

Part **(i) (b)** requires a formal proof. It will probably help to start by writing N as a product of prime factors, i.e. $N = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_k^{a_k}$ where $p_1, p_2, \cdots$ are distinct prime numbers and $a_1, a_2, \cdots$ are positive integers.

When looking for counterexamples, as in part (ii), the golden rule is to keep it simple: try to understand **why** the result is false, then use (in this case) low numbers.

The function defined in this question is called the *Euler totient* function. The value of $\mathrm{f}(N)$ is exactly equal to the number of integers less than or equal to N that are coprime to N (i.e. have no prime factors in common with N). Try it out — maybe starting with f(12) and f(20) and you'll begin to see how it works. For f(12), the numbers less than or equal to 12 that are coprime to 12 are 1,5,7,11; four of them, as expected.

Warm down

4 The following questions are taken from the first examinations set by the The University of Cambridge Local Examinations Syndicate in 1858. The Junior papers were for candidates "under 16 years of age" and the Senior papers were for candidates "under 18 years of age". The first two parts of the question below are from the Junior papers and the second two parts are from the Senior papers.

On another paper, candidates were asked to draw, from memory, a wheelbarrow turned upside down. They were allowed an hour to do so. You can do this if you want.

(i) Workmen can perform a certain labour in a week if they work 11 hours a day for 6 days; how many hours a day must they work to perform the same in the same time if they take half of Saturday as a holiday, but do a twelfth more work each hour?

We didn't get a very nice answer for this. Note that they worked long hours in those days!

(ii) The difference between two numbers is 3, and the difference of their cubes is 279, find the numbers.

(iii) Multiply $a^{\frac{3}{4}}x^{\frac{2}{3}} - b^{\frac{2}{5}}y^{\frac{1}{6}}$ by $a^{\frac{1}{4}}x^{\frac{1}{3}} - b^{\frac{3}{5}}y^{\frac{5}{6}}$.

Some not very nice numbers, again(!), for the cross terms.

(iv) Simplify $\dfrac{\sqrt{12+6\sqrt{3}}}{\sqrt{3}+1}$.

At last — a nice answer.