STEP Support Programme

Assignment 12

Warm-up

1 In this question, n denotes a positive integer.

(i) Show that $n(n+1)$ is divisible by 2.

Show that $n(n+1)(n+2)$ is divisible by 3.

(ii) Factorise completely $n^3 - n$, and deduce that it is divisible by 6.

(iii) Show that $n^5 - n^3$ is divisible by 24.

(iv) Show that $2^{2n} - 1$ is divisible by 3.

(v) Show that, if $n - 1$ is divisible by 3, then $n^3 - 1$ is divisible by 9.

For this part, factorising is not the easiest method; instead, you could try writing $n - 1 = 3k$ (where k is an integer).

Preparation

2 **(i)** Express the following as single fractions:

(a) $\dfrac{1}{(x-1)(x+2)} - \dfrac{1}{(x+1)(x+2)}$

(b) $\dfrac{m}{m+2} \times \dfrac{m-1}{m+1} + \dfrac{m}{m+2} \times \dfrac{2}{m+1} \times \dfrac{m-1}{m}$

(ii) I have a bag of sweets, which contains 9 mint imperials and 6 lemon sherbets. I take two sweets out of the bag without looking (one after the other) and eat them. What is the probability that I eat two sweets of the same flavour?

(iii) I have another bag of sweets with a apple sours and b blackcurrant chews.

(a) If I take one sweet at random, what is the probability that it is an apple sour?

(b) If I take three sweets one after the other and eat them what is the probability that they are all blackcurrant chews? (Don't attempt to simplify: leave your answer as the product of three fractions.)

(iv) Three children have a swimming lesson. Each child has a probability $\frac{1}{4}$, independently of the other two, of remembering to bring his or her goggles to the lesson. By considering the different possibilities find the probability that:

(a) at least one child has goggles;

(b) the first child in the queue has goggles;

(c) the middle child has not got goggles, but the other two have.

The STEP question

3 I am selling raffle tickets for £1 per ticket. In the queue for tickets, there are m people each with a single £1 coin and n people each with a single £2 coin. Each person in the queue wants to buy a single raffle ticket and each arrangement of people in the queue is equally likely to occur. Initially, I have no coins and a large supply of tickets. I stop selling tickets if I cannot give the required change.

(i) In the case $n = 1$ and $m \geqslant 1$, find the probability that I am able to sell one ticket to each person in the queue.

(ii) By considering the first three people in the queue, show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 2$ and $m \geqslant 2$ is $\dfrac{m-1}{m+1}$.

(iii) Show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 3$ and $m \geqslant 3$ is $\dfrac{m-2}{m+1}$.

Discussion

In parts (ii) and (iii) the answers are given, so you must **carefully** justify your answers. Set your work out so that the different possibilities for the queues appear in some logical order.

For part (iii), you will need to consider more than the first 3 people. You will need to persevere: you end up having to add up four or five long fractions (you can do some cancelling before you start adding).

You might have noticed a pattern in these three results, from which you might conjecture that the probability of being able to sell one ticket to each person in the general case $m \geqslant n$ is $\dfrac{m+1-n}{m+1}$. This turns out to be correct, though the proof is significantly more difficult than these special cases.

Warm down

4 A Rabbi, an Imam and a Bishop were having a chat. There was a knock at the door, and three bell ringers entered the room. After introductions, the Imam asked the Bishop how old the bell ringers were.

"Well," the Bishop said, knowing the Imam had a penchant for numerical puzzles, "if you multiplied their three ages together, you'd get 2,450. But if you added them, you'd get twice your age."

"Hmm," the Imam muttered, after several moments' thought. "I haven't enough information to solve that."

"It may help, my dear Imam," offered the Rabbi, "to know that I am older than anyone else here in the room."

"Yes, indeed it would," replied the Imam. "Now I know their ages."

The question is: How old is the Rabbi?

You may assume that all ages are integers.

This rather subtle puzzle is intended mainly for amusement, but also to initiate a quick discussion about the peculiarly English pursuit of Change Ringing. The bells are numbered from lightest (and highest pitched) to heaviest such as 1, 2, 3, 4, 5, 6. The lightest bell is called the Treble and the heaviest the Tenor. When they ring in descending order it is known as "rounds".

A "method" on six bells is a sequence of the form (......)(......)(......)(......) where the dots in each bracket represent one bell being rung and within each bracket none of the bells is repeated. The ordering or permutation within each bracket is different; no permutation is repeated.

For example, he first two brackets (changes) of "Plain Bob Minor" are (2 1 4 3 6 5) and (2 4 1 6 3 5).

There is a further rule: any bell can only change position by at most one place from one bracket to the next (each stroke). This is a rather practical rule. Bells can be a ton or more in weight and once they get going it is hard to change their rhythm significantly.

If all the possible permutations are rung (once and once only!) then this is known as an "extent" or "full peel". The length of an extent on 5 bells is $5! = 120$ and takes about 5 minutes to ring. An extent on 8 bells has only ever been rung once which was in 1963 in Loughborough: it took just under 18 hours. An extent on the 12 bells of St Mary's in Cambridge would take over 30 years to ring.