STEP Support Programme

Assignment 13

Warm-up

You probably already know that for a graph with gradient $\dfrac{\mathrm{d}y}{\mathrm{d}x}$:

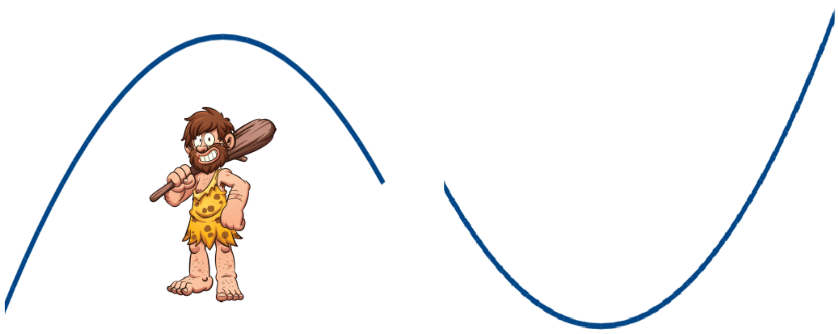
- if $\dfrac{\mathrm{d}y}{\mathrm{d}x} > 0$ then the graph is *increasing*;
- if $\dfrac{\mathrm{d}y}{\mathrm{d}x} < 0$ then the graph is *decreasing*.

The sign of the second derivative also tells us something about the shape of a graph:

- if $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ then the graph is *concave*, which means that the gradient of the curve is decreasing — it is bending downwards;
- if $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ then the graph is *convex*, which means that the gradient of the curve is increasing — it is bending upwards.

A convex curve is like a smile.

The picture below shows the difference. If you get muddled up, remember that a **cave**man can live in a con**cave** graph.

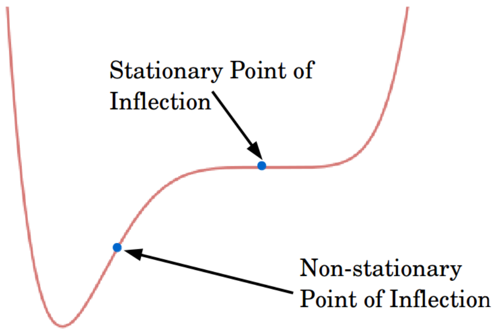


Concave
Gradient is decreasing

Convex
Gradient is increasing

A *point of inflection* is a point at which the graph changes from being concave to being convex or vice versa. At the point of inflection $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$; in addition, the sign of the second derivative must be different on either of the point. Points of inflection can be stationary (if $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ as well) or non-stationary.

The diagram below shows the difference between a stationary and a non-stationary point of inflection. In each case you can see that the "bend" of the curve changes on each side of the point.



1 **(i)** Find a range of values of x for which the graph $y = x^4 - 6x^2 + 9$ is concave.

(ii) For the graph $y = x^3 - 2x^2 - 3x$, find the point where $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$.

(iii) For the graph $y = x^4 - 2x^3$, find the stationary points and the points where $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$. Sketch $y = x^4 - 2x^3$.

(iv) For the graph $y = (x-1)^4$, find the point where $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$. By considering the shape of the graph show that this is not a point of inflection.

Note that **if** we have a point of inflection **then** $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at this point, but **if** $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at some point it does not necessarily mean that this point is a point of inflection. In other words, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ is a *necessary* condition for a point to be a point of inflection, but it is not a *sufficient* condition.

Discussion

We stated that

- if $\dfrac{\mathrm{d}y}{\mathrm{d}x} > 0$ then the graph is *increasing*

and this is correct. *However*, it is not the definition of an increasing graph; it is a sufficient condition for the graph to be increasing, but not a necessary condition. In other words, it is 'if ... then' but not 'if and only if ... '.

The usual mathematical definition is:

- the graph $y = \mathrm{f}(x)$ is *increasing* if and only if $\mathrm{f}(x+h) \geqslant \mathrm{f}(x)$ for $h > 0$.

Note that the inequality is not strict, so the graph $y = 1$ is increasing according to this definition. This seems peculiar, but it turns out to be convenient.[1]

If we replace $\geqslant$ with $>$, we say that the graph is *strictly increasing*.

Note also that this definition works even for functions (such as flight of steps, or the floor function) which do not have a derivative at all points.

If you are trying to classify stationary points, and you have one where $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ then there are various ways to proceed.

- You can look at the sign of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ on either side of the stationary point. If the sign of the gradient is the same both sides then you have a point of inflection. If the gradient is negative to the left of the stationary point and positive to the right of the stationary point then you have a minimum.
- You could instead look at the sign of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ either side of the stationary point. If $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ both sides then you have a minimum and if $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ then you have a maximum. If $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ changes sign either side of the stationary point then there is a point of inflection.

There are other techniques you can use, and one that might be very useful is to consider what you already know about the graph (location and nature of other stationary points, what happens as $x \to \pm\infty$ etc.) to deduce the nature of the stationary point in question.

[1] For example, a (differentiable) function is increasing if and only if $\frac{\mathrm{d}y}{\mathrm{d}x} \geqslant 0$, which is convenient. On the other hand, it is not that case that a (differentiable) function is strictly increasing if and only if $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$; think of $y = x^3$, which is certainly strictly increasing (it gets bigger and bigger!), but $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ (so is not strictly positive) at $x = 0$.

Preparation

2 **(i)** Consider the graph $y = x^3 - 3x + 2$. Sketch the graph by:

(a) factorising y and hence finding the roots, **and**
(b) finding the coordinates of the turning points, **and**
(c) finding where the curve crosses the y-axis.

How many distinct roots of the equation $x^3 - 3x + 2 = 0$ are there?

Note that the graph has a "repeated" root at $x = 1$. This occurs when a turning point is located on the x-axis. When asked for *distinct* roots, we count a repeated root only once.

(ii) By considering each of the following graphs as a transformation of $y = x^3 - 3x + 2$, sketch it (showing the coordinates of the turning points and y-intercept) and state how many distinct roots there are.

Note that you are not asked to find the *values* of the roots.

(a) $y = x^3 - 3x$
(b) $y = x^3 - 3x + 4$
(c) $y = x^3 - 3x - 4$

State the values of k for which the equation $x^3 - 3x + k = 0$ has **(A)** 2 distinct roots and **(B)** 3 distinct roots.

Use the idea of translating the original graph to help you. For part **(B)** there is a range of values of k.

(iii) Consider the graph of $y = 3x^4 + 4x^3 - 6x^2 - 12x + 5$.

(a) Find the coordinates of the stationary points.
(b) Find the points where $\dfrac{d^2y}{dx^2} = 0$. State whether each one is stationary or non-stationary.
(c) Sketch the graph.
(d) Write down the value of k for which the equation $3x^4 + 4x^3 - 6x^2 - 12x + k = 0$ has only one root.

The STEP question

3 **(i)** Sketch the curve $y = x^4 - 6x^2 + 9$ giving the coordinates of the stationary points.

Let n be the number of distinct real values of x for which

$$x^4 - 6x^2 + b = 0.$$

State the values of b, if any, for which (a) $n = 0$; (b) $n = 1$; (c) $n = 2$; (d) $n = 3$; (e) $n = 4$.

(ii) For which values of a does the curve $y = x^4 - 6x^2 + ax + b$ have a point at which both $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$?

For these values of a, find the number of distinct real values of x for which

$$x^4 - 6x^2 + ax + b = 0\,,$$

in the different cases that arise according to the value of b.

(iii) Sketch the curve $y = x^4 - 6x^2 + ax$ in the case $a > 8$.

For part **(ii)** you could sketch $y = x^4 - 6x^2 + ax$ for the particular values of a.

Warm down

I have n £1 coins and p £5 and q £10 notes (at least one of each). As it happens, the total amount of money would be unchanged if instead I had p £1, q £5 notes, and n £10 notes. I want to investigate the possible values of (n, p, q).

Consider the case $n = 6$.

(a) There is one very obvious solution. What is it?

(b) Write down an equation relating p and q and sketch the graph of this equation in the p-q plane (i.e. with p and q on the axes).

(c) Show that if $(6, p, q)$ is a solution, then $(6, p + 5k, q - 4k)$, where k is an integer, is another solution. [You may like to think whether every solution is of this form.]

(d) Find the three solutions.

How is the situation different if $p = 6$ (instead of $n = 6$)?