STEP Support Programme

Assignment 15

Warm-up

1 **(i)** Simplify $1 - \dfrac{1}{1 - \dfrac{1}{1 - \frac{1}{x}}}$.

(ii) Show that $x = 15$ is a root of the equation $x^4 - 18x^3 + 35x^2 + 180x - 450 = 0$ (you should be able to do this without a calculator, if you do some factorisation). Find all the roots of the equation.

(iii) The notation $\displaystyle\prod_{r=1}^{n} \mathrm{f}(r)$ denotes the product $\mathrm{f}(1) \times \mathrm{f}(2) \times \mathrm{f}(3) \times \cdots \times \mathrm{f}(n)$.

(a) Evaluate $\displaystyle\prod_{r=1}^{4} r$.

(b) Simplify $\displaystyle\prod_{r=1}^{n} \frac{r}{r+1}$.

(c) Simplify $\displaystyle\prod_{r=1}^{n} \frac{\mathrm{g}(r)}{\mathrm{g}(r-1)}$, given that $\mathrm{g}(r-1) \neq 0$ for $r = 1,\ 2,\ \ldots,\ n$.

For parts **(b)** and **(c)** you may like to write out the first few terms (and the last one!)

Preparation

2 Sequences can behave in different ways.

- Formally, we say that the sequence $a_1, a_2, \ldots$ *converges* if there is a number ℓ such that $a_n \to \ell$ as $n \to \infty$, i.e. the terms of the sequence get closer and closer to some fixed number ℓ. For example, the following sequences converge:

$$\frac{1}{2},\ \frac{3}{4},\ \frac{7}{8},\ \frac{15}{16},\ \ldots \qquad (\ell = 1)$$

$$4,\ -2,\ 1,\ -\frac{1}{2},\ \frac{1}{4},\ \ldots \qquad (\ell = 0)$$

- We say that a sequence *diverges* if it doesn't converge; note that this allows the possibility of a sequence diverging even though its terms don't get big. For example, the following sequences diverge:

$$1,\ 3,\ 9,\ 27,\ \ldots$$

$$a_n = n^{\text{th}} \text{ digit of } \pi \quad \text{(as far as is known!)}$$

- We say that a sequence $a_1, a_2, \ldots$ is *periodic* if $a_n = a_{n+k}$ for some fixed k and all n. i.e. the terms repeat the same values over and over. The integer k is called the *period* of the sequence. For example, the following sequences are periodic:

$$1,\ -1,\ 1,\ -1,\ 1, \ldots$$

$$1,\ 2,\ 3,\ 1,\ 2,\ 3, \ldots$$

  The first of these has period 2 and the second has period 3.[1]

- A sequence $a_1, a_2, \ldots$ is said to be *constant*, if there is a number a such that $a_n = a$ for all n, i.e. the terms are the same. The only periodic sequence that converges is a constant sequence.

Whilst calculators are not allowed in STEP examinations, for some parts of this question, a calculator (or even spreadsheet) and decimal approximations may be appropriate.

(i) Find the first 6 terms of the following sequences and describe (you don't have to give a proof) their behaviour.

(a) $u_1 = 1$ and $u_{n+1} = u_n^2 - 3$

(b) $u_1 = 1$ and $u_{n+1} = 6 - \dfrac{4}{u_n}$

(c) $u_1 = 2$ and $u_{n+1} = \frac{1}{4}(u_n^2 + 2)$

(d) $u_1 = 5$ and $u_{n+1} = \frac{1}{4}(u_n^2 + 2)$

For the sequences above that converge to a limit, find this limit by setting $u_n = u_{n+1} = l$ and solving for l (i.e. replace both u_n and u_{n+1} with l and then find l).

[1] Sometimes, a sequence is said to be periodic if the terms repeat themselves after the N^{th} term, for some integer N, for example $3, 6, 2, 4, 5, 1, 5, 1, 5, 1, 5, 1, \ldots$

(ii) Write down the first 10 terms of the following sequence, and describe its behaviour.

$$u_1 = 4 \quad \text{and} \quad u_2 = 1 \quad \text{and} \quad u_{n+2} = u_{n+1} - u_n$$

(iii) Consider the sequence $u_1 = 2$ and $u_{n+1} = u_n^2 - b$.

(a) Write u_2 and u_3 in terms of b.

(b) Find the possible values of b for which $u_3 = u_1$.

(c) Describe the behaviour of the sequence for each of these values of b.

(iv) Find the first 5 terms of the convergent sequence defined by

$$u_1 = 1\,, \qquad \text{and} \qquad u_{n+1} = \frac{1}{2}\left(\frac{2}{u_n} + u_n\right) \quad \text{for } n \geqslant 1.$$

By setting $u_n = u_{n+1} = l$ find the limit ℓ of this sequence.[2]

(v) Find the limit of the sequence defined by

$$u_1 = 1\,, \qquad \text{and} \qquad u_{n+1} = \frac{1}{3}\left(\frac{a}{u_n^2} + 2u_n\right) \quad \text{for } n \geqslant 1.$$

By choosing a suitable value for a, use this sequence to find $\sqrt[3]{7}$.

The STEP question

No calculators or spreadsheets for this question please!

3 The sequence of real numbers u_1, u_2, u_3, ... is defined by

$$u_1 = 2\,, \qquad \text{and} \qquad u_{n+1} = k - \frac{36}{u_n} \quad \text{for } n \geqslant 1, \qquad (*)$$

where k is a constant.

(i) Determine the values of k for which the sequence $(*)$ is:

(a) constant;

(b) periodic with period 2;

(c) periodic with period 4.

(ii) In the case $k = 37$, show that $u_n \geqslant 2$ for all n. Given that in this case the sequence $(*)$ converges to a limit ℓ, find the value of ℓ.

[2] You should find that u_5 is accurate to at least 9 places of decimals!

Discussion

This is quite a daunting question, involving many different ideas.

Note that for parts **(i)(b) and (i)(c)** you will have to work out u_{n+2} and u_{n+4} (respectively) in terms of u_n. Then it is sufficient to solve $u_3 = u_1 = 2$ and $u_5 = u_1 = 2$ in the two cases (why?). The former will give you a quadratic to solve, and one of the answers should not surprise you. The latter will give you a quartic, which you can solve because two solutions will already be known to you.

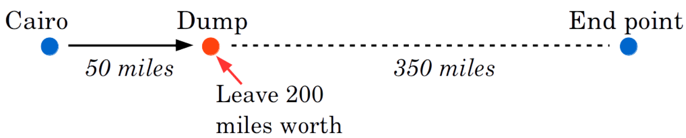
For the first part of **(ii)**, you might want to use the method of induction, if you know it. If those don't know it, the argument runs along these lines: if $u_n > 2$ then the recurrence relation shows that $u_{n+1} > 2$; since $u_1 > 2$, we have $u_2 > 2$ and hence $u_3 > 2$ and hence $u_4 > 2$, etc for ever.

Warm down

4 **(i)** I want to drive 400 miles across the desert from Cairo. My car holds only enough petrol for 300 miles. At any stage of my journey, I can drain petrol from my car and dump it in a can by the path, to be picked up later, while I return to Cairo to fill up again. I can return to Cairo and fill up as many times as I want, but Health and Safety regulations prevent me from carrying any extra petrol in cans in the car.

Show that I can complete my journey with just one petrol dump and one return to Cairo, driving a total of 600 miles.

Your answer could be in the form of a series of pictures showing each of the individual parts (between filling up and dumping) of my journey. The first picture could look like this:



Don't forget that you must leave enough petrol to get back to Cairo if you want to refuel. The above diagram is not a good first step.

(ii) Suppose instead that I want to drive 460 miles from Cairo. Show that by establishing a dump 60 miles from Cairo and another dump further on, I can complete my journey driving a total of 900 miles.