STEP Support Programme

Assignment 17

Warm-up

1 This question is about *modular arithmetic* (and so is the warm-down); a very simple idea, but a very important tool in the theory of numbers.

We say that
$$N \bmod a = n$$
if the remainder when N is divided by a is n. For example
$$10 \bmod 3 = 1 \qquad \text{and} \qquad 25 \bmod 5 = 0\,,$$
since $10 \div 3 = 3$ remainder 1, and $25 \div 5 = 5$ with no remainder.

The possible values of N mod 3 are 1, 2 or 0 (if N is divisible by 3). This means that N can be written in the form
$$N = \begin{cases} 3m & \text{if } N \bmod 3 = 0 \\ 3m+1 & \text{if } N \bmod 3 = 1 \\ 3m+2 & \text{if } N \bmod 3 = 2 \end{cases}$$
where m is some integer.

More generally,
$$N \bmod a = n \iff N = am + n \qquad (*)$$
for some integer m.

(i) Use $(*)$ to prove that if $N_1 \bmod a = n_1$ and $N_2 \bmod a = n_2$ then
$$(N_1 + N_2) \bmod a = (n_1 + n_2) \bmod a\,.$$
and
$$N_1 N_2 \bmod a = n_1 n_2 \bmod a\,.$$
These two results are required over and over again when using modular arithmetic.

Start by writing $N_1 = am_1 + n_1$ and similarly for N_2. A bit of care is needed as it is not necessarily true that $(N_1 + N_2) \bmod a = (n_1 + n_2)$ (because $n_1 + n_2$ might be greater than a — in which case, it can be written as $a + (n_1 + n_2) \bmod a$).

It is often a good idea to use a numerical example to make sure you fully understand a new idea. So we could consider $22 \bmod 5 = 2$ and $8 \bmod 5 = 3$. We can then see that $(22 \times 8) \bmod 5 = 176 \bmod 5 = 1$ and $(2 \times 3) \bmod 5 = 6 \bmod 5 = 1$.

Note that a numerical example is **not** a proof, but it is a useful sanity check.

If $x \bmod a$ is the same as $y \bmod a$ then we can say that x and y are *congruent mod a*. We write this as:

$$x \equiv y \pmod a$$

(note the brackets). For example,

$$16 \equiv 31 \pmod 5 \qquad \text{and} \qquad 14 \equiv -1 \pmod 3.$$

In the first case it is clear that both 16 and 31 leave the same remainder when they are divided by 5. In the second case, note that $14 = 3 \times 4 + \mathbf{2}$ and $-1 = 3 \times -1 + \mathbf{2}$.

Two numbers are congruent mod a if their difference is equal to 0 mod a i.e.

$$x \equiv y \pmod a \iff (x - y) \bmod a = 0.$$

Note that $\big(14 - (-1)\big) \bmod 3 = 0$.

So if $N_1 \equiv n_1 \bmod a$ and $N_2 \equiv n_2 \bmod a$, then we can write

$$N_1 + N_2 \equiv n_1 + n_2 \pmod a \qquad \text{and} \qquad N_1 N_2 \equiv n_1 n_2 \pmod a. \qquad (**)$$

(ii) Noting that $10 \bmod 3 = 1$, show that $(10a + b) \bmod 3 = (a + b) \bmod 3$, where a and b are integers.

Deduce that a decimal number between 0 and 99 with digits ab is divisible by 3 if and only if $a + b \equiv 0 \pmod 3$.

You have to use the results of part (i) quite a few times to do the first part; or you can use the notation of $(**)$.

By considering $10^n \bmod 3$, devise a test to determine whether an n-digit decimal number with digits $a_1 a_2 \dots a_n$ is divisible by 3.

(iii) Devise a test to determine whether a 4-digit decimal number with digits $abcd$ is divisible by 11.

Preparation

2 **(i)** Solve the following set of simultaneous equations.

$$\begin{aligned} w + x + y + z &= 1 \\ w - x + y - z &= 0 \\ 4w + 3x + 2y + 3z &= 3 \\ 4w - 3x + 2y - 9z &= -1 \end{aligned}$$

(ii) Evaluate the following:

(a) $\displaystyle\sum_{i=3}^{5} \frac{i}{i-1}$

(b) $\displaystyle\sum_{r=-2}^{2} r^2$

(iii) Show that $\dfrac{1}{r(r+1)} = \dfrac{1}{r} - \dfrac{1}{r+1}$. Hence find an expression for $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)}$.

Check your answer by taking $n = 3$.

Find $\displaystyle\sum_{r=100}^{200} \frac{1}{r(r+1)}$

(iv) By setting $x = 0$ and $x = 1$ in turn, find a and b such that

$$\frac{1}{x^2 + 3x + 2} \equiv \frac{a}{x+1} + \frac{b}{x+2}$$

The STEP question

3 It is given that $\displaystyle\sum_{r=-1}^{n} r^2$ can be written in the form $pn^3 + qn^2 + rn + s$, where p, q, r and s are numbers. By setting $n = -1$, 0, 1 and 2, obtain four equations that must be satisfied by p, q, r and s and hence show that

$$\sum_{r=0}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)\ .$$

Given that $\displaystyle\sum_{r=-2}^{n} r^3$ can be written in the form $an^4 + bn^3 + cn^2 + dn + e$, show similarly that

$$\sum_{r=0}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2\ .$$

Warm down

4 A certain sort of divisibility question used to come up on scholarship papers, intended perhaps as an exercise in the use of proof by mathematical induction; though often they could be more easily and more enjoyably done using modular arithmetic. Question (iii) below is a good example, which occurred on a STEP paper in 1987 (there has not been an example since). Parts (i) and (ii) are supposed to give you some ideas for tackling part (iii). Of course, the basic idea for showing that a number N is divisible by a (say) is to show that $N \bmod a = 0$, but you sometimes have to be quite ingenious.

(i) Show that $3 \times 2^{2n} + 2 \times 3^{2n}$ is divisible by 5.

You could either use $3 \equiv -2 \pmod 5$ or $2^{2n} \equiv 4^n \equiv (-1)^n \pmod{5}$ with a similar result for 3^{2n}.

(ii) Show that, if n is odd, then $2^n + 5^n + 56$ is divisible by 3. Show also that it is divisible by 21.

Show further it is divisible by 63 if $n \bmod 3 = 0$ (and n is odd).

Note that if n is odd and divisible by 3, then it can be written as $3m$ *where m is odd.*

In fact $2^n + 5^n + 56$ is divisible by 63 whenever n is odd, but a bit more thought is needed to get the factor of 9 if n is not divisible by 3. If you feel brave you might start by thinking about the statement 'if n is odd and divisible by 3, you can write it as $3m$ where m is odd'. How can we write n if it is odd but not divisible by 3? The way I did it required two different forms depending on the value of $n \bmod 3$.

(iii) Show that $2^{3n+1} + 3 \times 5^{2n+1}$ is divisible by 17.