STEP Support Programme

Assignment 18

Warm-up

1 When sketching curves, the things to consider are intercepts with the x and y axes, turning points and asymptotes (i.e. what happens as $x \to \pm\infty$ or $y \to \pm\infty$). Often, you can sketch the curve without using all this information (though it is good to have it all). You may find it useful to think about transformations (translations in particular).

(i) Sketch the curve $y = \dfrac{1}{x-1}$.

(ii) Sketch the curve $y = \dfrac{x}{x-1}$.

What happens as $x \to \pm\infty$? It may be helpful to write $\dfrac{x}{x-1}$ in the form $a + \dfrac{b}{x-1}$.

(iii) Sketch the curve $y = \dfrac{x^2}{x-1}$.

Here you can divide x^2 by $x - 1$ to enable you to write $y = \dfrac{x^2}{x-1}$ in the form $y = ax + b + \dfrac{c}{x-1}$. You can then see that as $x \to \pm\infty$ then $y \approx ax + b$. Be careful with your long division and double check that your answer still gives $y = \dfrac{x^2}{x-1}$.

You may need the derivative of $(x-1)^{-1}$ which is $-(x-1)^{-2}$, by the chain rule.

(iv) Sketch the curve $y = \dfrac{1}{x-1} + \dfrac{1}{x+1}$.

A small check on the general shape is to see whether the curve approaches the horizontal axis (as $x \to \pm\infty$) from above or from below. It may be helpful to consider the sign of the gradient.

Preparation

2 **(i)** Solve the quadratic equation $(a+2)x^2 - 2x - a = 0$.

(ii) For what values of x is $-7x - 7 < 0$? Show that $3x^3 - x^2 - 7x - 7 < 0$ when $x = -\frac{7}{9}$.

You don't have to calculate $(-\frac{7}{9})^3$.

(iii) If $a/b > 0$, what can be said about a and b?

Let $y = \dfrac{x}{x-1}$.

If $y > 0$ and $x > 0$, show that $x > 1$. If in addition $y > x$, show that $x < 2$.

You could use the sketch from Q1(ii), but it is better from the point of view of the STEP question to do it without a sketch: the argument is just a few lines.

The STEP question

3 The numbers a and b, where $b > a \geqslant 0$, are such that

$$\int_a^b x^2 \, \mathrm{d}x = \left(\int_a^b x \, \mathrm{d}x \right)^2 .$$

(i) In the case $a = 0$ and $b > 0$, find the value of b.

(ii) In the case $a = 1$, show that b satisfies

$$3b^3 - b^2 - 7b - 7 = 0 .$$

Show further, with the help of a sketch, that there is only one (real) value of b that satisfies this equation and that it lies between 2 and 3.

(iii) Show that $3p^2 + q^2 = 3p^2 q$, where $p = b + a$ and $q = b - a$, and express p^2 in terms of q. Deduce that $1 < b - a \leqslant \frac{4}{3}$.

Discussion

In part **(ii)** you may get a quartic in b. If so, can you spot an "obvious" root?

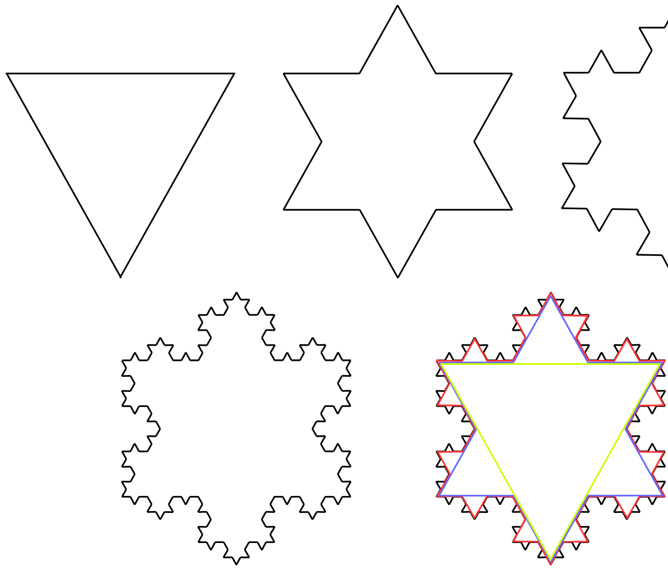
In part **(iii)** you have to return to the integrals given in the very first line of the question and try to write the expression you get in terms of p and q.

Warm down

4 This question is about a *fractal* curve.[1,2] The diagrams below show how our fractal is created. We start with an equilateral triangle. Then each side is divided into three equal parts and the middle section is replaced by two sides of an equilateral triangle. There are now 12 edges.

At the next step each of these edges is divided into three equal parts and again the middle section is replaced by two sides of an equilateral triangle.

The process is then repeated over and over again. The first four diagrams below show the original triangle and the first three iterations. The final diagram just shows these iterations superposed.[3]



(i) How many edges are there in the second iteration? Find an expression for the number of edges in the n^{th} iteration.

(ii) Let each side of the original triangle have length 1. How long is each of the 12 edges in the first iteration? Write down an expression for the length of each edge in the n^{th} iteration.

(iii) Using your answers to parts **(i)** and **(ii)** find an expression for the total length of the curve in the n^{th} iteration. What happens to this total length as $n \to \infty$?

(iv) Let the area of the original triangle be A. By considering the areas of each of the 3 small triangles added in the first iteration, find the total area of the first iteration of the curve. Similarly, find an expression for the total area of the second iteration.

Obtain an expression for the total area of the n^{th} iteration as a sum of the form $A + \frac{1}{3}A(1 + r + r^2 + \cdots + r^{n-1})$ for some number r which you should find. What happens to the total area as $n \to \infty$?

[1] Fractals are infinitely complex patterns that are self-similar across different scales. They are created by repeating a simple process over and over.

[2] You may have heard Elsa singing about "Frozen fractals all around".

[3] This fractal is called the Koch Snowflake.