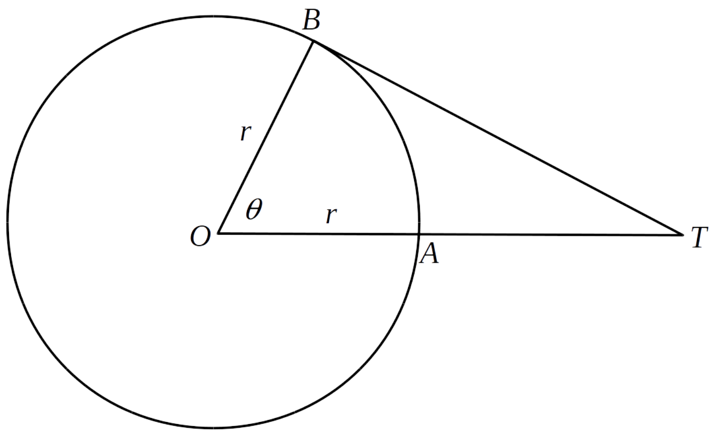
STEP Support Programme

Assignment 19

Warm-up

1 The picture below shows a circle of radius r with centre O and a right-angled triangle OBT.



(i) By considering the areas of the triangle OBT, the sector OBA and the triangle OBA, show that:

$$\frac{1}{\cos\theta} > \frac{\theta}{\sin\theta} > 1\,.$$

(ii) We know that $\cos\theta \to 1$ as $\theta \to 0$. We can write this result in the form

$$\lim_{\theta\to 0}\left(\frac{1}{\cos\theta}\right) = 1\,.$$

Using part **(i)**, find the value of $\lim_{\theta\to 0}\left(\dfrac{\theta}{\sin\theta}\right)$. (You should justify your answer briefly). What is therefore the approximate value of $\sin\theta$ for small values of θ?

(iii) Use a similar argument, starting from part **(i)**, to show that:

$$\lim_{\theta\to 0}\left(\frac{\theta}{\tan\theta}\right) = 1$$

and give an approximation for $\tan\theta$ when θ is small.

(iv) Use $\cos^2\theta + \sin^2\theta = 1$ and the binomial expansion to show that $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ when θ is small.

One form of the familiar binomial expansion is:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

where n is a positive integer. In this form, the result also holds when n is not a positive integer: n can be negative, for example, or any non-integer (such as $\frac{1}{2}$). It can even be a complex number! When n is not a positive integer, we need the condition $|x| < 1$ in order for this series to *converge*; otherwise the terms get bigger and bigger in a meaningless way.

(v) It is very important that you realise that these limits and "small angle approximations" are only valid when θ is in **Radians!** Where in your proofs did you assume that θ is measured in radians?

Preparation

2 **(i)** Find an expression for the distance between the point P with coordinates $(2\,,-2)$ and point X with coordinates $(x\,,y)$.

(ii) Point Q has coordinates $(4\,,0)$. By considering the distances PX and QX, find the equation of the line of points equidistant from P and Q.

(iii) Find the values of a and b for which the following equations describe the same line:

$$x + ay = 2$$
$$ax + 4y = b$$

Note that $x + y = 1$ and $2x + 2y = 2$ describe the same line even though the equations are not identical.

(iv) Given that:

$$5 = \frac{2px - y}{1 - p}\,,$$

find p in terms of x and y.

(v) Show that the equation $xy + 2y + 3x - 54 = 0$ can be written in the form:

$$(x+2)(y+3) = 60\,.$$

Given that x and y are both positive integers, find the possible values of x and y.

The STEP question

3 **(i)** The point A has coordinates $(5\,,16)$ and the point B has coordinates $(-4\,,4)$. The variable point P has coordinates $(x\,,y)$ and moves on a path such that $AP = 2BP$. Show that the Cartesian equation of the path of P is

$$(x+7)^2 + y^2 = 100\,.$$

(ii) The point C has coordinates $(a\,,0)$ and the point D has coordinates $(b\,,0)$, where $a \neq b$. The variable point Q moves on a path such that

$$QC = k \times QD\,,$$

where $k > 1$. Given that the path of Q is the same as the path of P, show that

$$\frac{a+7}{b+7} = \frac{a^2+51}{b^2+51}\,.$$

Show further that $(a+7)(b+7) = 100$.

Discussion

A note on algebra.

There is quite a bit of algebraic manipulation in this question. Don't be afraid of it! BUT: do write out all the stages carefully, rather than doing them in your head or on scraps of paper, checking each line before going on to the next line.

For any number k, the equation $QC = k \times QD$ in part (ii) describes a circle (as you should see from your calculations).

The very last line of the question looks surprising. But in fact it follows from the previous displayed equation after a bit of algebra.

The circle given by $QC = k \times QD$ is called a *circle of Apollonius*. (It is the straight line perpendicular bisector of CD if $k = 1$, which can be thought of as a circle of infinite radius.) Apollonius of Perga (roughly 262 BC — 190 BC) was a Greek geometer and astronomer. He asked the question: if two ships start a distance d apart and travel in straight lines at constant speeds u and v, where do they meet? You can probably see the connection between his question and his circles.

Warm down

4 **(i)** I place three coins in a bag. One coin is normal, with a head on one side and a tail on the other. One coin has heads on both sides and the other has tails on both sides. I pick one coin from the bag at random and look at one side of it at random, and it is a head. What is the probability that there is a head on the other side?

One way of tackling this is to work out all the (equally likely) possibilities. Listing all the possibilities is not "cheating" and can sometimes be the most efficient way to solve a problem. It becomes less efficient the more possibilities there are.

(ii) I am about to throw three fair dice. What is the probability of three sixes? What is the probability of one six?

My friend offers to give me £1 if I throw no sixes provided I give her £1 if I throw one six, £2 if I throw two sixes and £3 if I throw three sixes. Should I accept this offer?