STEP Support Programme

Assignment 20

Warm-up

1 **(i)** By using the definition:

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h} \qquad (*)$$

find the derivative of $\sin x$.

You will need the result $\sin(A+B) = \sin A \cos B + \sin B \cos A$ (see Assignment 10 for a proof) and the small angle approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ (which hold when θ is in radians — see Assignment 19 for a proof).

Find also the derivative of $\cos x$.

(ii) Use the definition $(*)$ to find the derivative of $\ln x$.[1]

You will need to use the series expansion[2]:

$$\ln(1+t) = t - \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 - \tfrac{1}{4}t^4 + \cdots \quad \text{for } -1 < t < 1\,.$$

Note that the argument of the logarithm must be in the form "$1+t$" before you can use the expansion, and your "t" must be in the range $-1 < t < 1$.

(iii) Find a number k such that, when t is small, $\sqrt{1+t} \approx 1 + kt$, where the approximation means that t^2 and smaller terms are ignored. Hence find the derivative of $\sqrt{x}$. Find also the derivative of $x^{-\frac{1}{2}}$.

You don't need the binomial expansion for either of these, but you may like to use it for the second.

[1] $\ln x$ is the "natural logarithm", i.e. logarithm to the base e where $\mathrm{e} = 2.718\ldots$. $\ln x$ obeys all the usual log laws so, for example, $\ln(AB) = \ln A + \ln B$.

[2] Knowledge of this (and other Maclaurin series) is not needed for STEP I.

Preparation

2 Proof by Induction

Proof by induction is part of the STEP I specification, even though it only appears on Further Mathematics specifications.

The following shows an example of how to prove a conjecture by induction.

Question: Prove that $11^n - 4^n$ is divisible by 7 for integers $n \geqslant 1$.

When $n = 1$ (the 'base case') we have $11^1 - 4^1 = 7$, so the result is true when $n = 1$.

Assume the conjecture is true when $n = k$, so we have $11^k - 4^k = 7M$ for some integer M.

Now consider the case $n = k + 1$. We are required to prove (RTP) $11^{k+1} - 4^{k+1} = 7P$ for some integer P.

You should expect to use the $n = k$ case at some point (otherwise there was not a lot of point in assuming it was true!).

$$\begin{aligned}
11^{k+1} - 4^{k+1} &= 11 \times 11^k - 4 \times 4^k \\
&= 11 \times \left(7M + 4^k\right) - 4 \times 4^k \text{ using the } n = k \text{ case} \\
&= 11 \times 7M + 11 \times 4^k - 4 \times 4^k \\
&= 11 \times 7M + 7 \times 4^k \\
&= 7 \times \left(11M + 4^k\right) \\
&= 7P \text{ where } P \text{ is an integer.}
\end{aligned}$$

Therefore, if it is true for $n = k$ it is true for $n = k + 1$.

But we know it is true for $n = 1$, and by taking $k = 1$, we find that it is true for $n = 2$. Now, taking $k = 2$, we find that it is true for $n = 3$. We can go on like this indefinitely[3] to show that it is true for any integer value of n.

Use induction to prove the following:

(a) $4^n + 6n - 1$ is divisible by 9 for any positive integer n.

(b) $\displaystyle\sum_{i=1}^{n} i^3 = \tfrac{1}{4}n^2(n+1)^2$ where $n \geqslant 1$.

It is very helpful when considering the $n = k + 1$ case to write down what you are required to prove ('RTP') so that you know what you are aiming for. For part **(b)** factorisation is your friend — don't expand first and then have to factorise a quartic! (You can if you really want to, but it's not the most efficient use of time).

[3]For the (old) MEI A-levels you were supposed to write (for the final 2 marks) something like: "But this is the given result with $k+1$ replacing k. Therefore if it is true for k it is true for $k+1$. Since it is true for $n = 1$, it is true for $n = 1, 2, 3, \ldots$ and so true for all positive integers." This is a good thing to write: it shows that you understand the method (or that you can learn by heart a few sentences).

The STEP Question

3 The Fibonacci numbers F_n are defined by the conditions $F_0 = 0$, $F_1 = 1$ and

$$F_{n+1} = F_n + F_{n-1}$$

for all $n \geqslant 1$. Show that $F_2 = 1$, $F_3 = 2$, $F_4 = 3$ and compute F_5, F_6 and F_7.

Compute $F_{n+1}F_{n-1} - F_n^2$ for a few values of n; guess a general formula and prove it by induction, or otherwise.

By induction on k, or otherwise, show that

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

for all positive integers n and k.

Discussion

Remember that for a "Show that" you must show enough working to fully justify your answer (a "Show that" needs as much working as a "Prove that"). For the last part you are using induction on k (or otherwise), so your first step might be to "assume true for $k = m$" and then you can "consider the case $k = m + 1$".

In each case you will need to use the *recurrence relation* for the Fibonacci numbers (i.e. $F_{n+1} = F_n + F_{n-1}$) at some point. Have a play around with different approaches before asking for help on the forum, the proofs are quite short but you will probably go down a couple of wrong alleys before finding one.

In some proofs by induction you will need the previous two cases (lets call them P_k and P_{k-1} where P_k is "Proposition when $n = k$") in order to prove P_{k+1}. If this is the case then you will need to show that two base cases are true.

In some proofs by induction you will need all of the previous cases $(P_1, P_2, \cdots, P_k)$ in order to prove P_{k+1}. This is called *Strong Induction*.

The question below is a STEP III question, but is quite a fun one![4]

4 $\triangle$ is an operation that takes polynomials in x to polynomials in x; that is, given any polynomial $\mathrm{h}(x)$, there is a polynomial called $\triangle\mathrm{h}(x)$ which is obtained from $\mathrm{h}(x)$ using the rules that define $\triangle$. These rules are as follows:

(i) $\triangle x = 1\,;$

(ii) $\triangle\big(\mathrm{f}(x) + \mathrm{g}(x)\big) = \triangle\mathrm{f}(x) + \triangle\mathrm{g}(x)$ for any polynomials $\mathrm{f}(x)$ and $\mathrm{g}(x)$;

(iii) $\triangle\big(\lambda\mathrm{f}(x)\big) = \lambda\triangle\mathrm{f}(x)$ for any constant λ and any polynomial $\mathrm{f}(x)$;

(iv) $\triangle\big(\mathrm{f}(x)\mathrm{g}(x)\big) = \mathrm{f}(x)\triangle\mathrm{g}(x) + \mathrm{g}(x)\triangle\mathrm{f}(x)$ for any polynomials $\mathrm{f}(x)$ and $\mathrm{g}(x)$.

Using these rules show that, if $\mathrm{f}(x)$ is a polynomial of degree zero (that is, a constant), then $\triangle\mathrm{f}(x) = 0$. Calculate $\triangle x^2$ and $\triangle x^3$.

Prove that $\triangle\mathrm{h}(x) \equiv \dfrac{\mathrm{dh}(x)}{\mathrm{d}x}$ for any polynomial $\mathrm{h}(x)$. You should make it clear whenever you use one of the above rules in your proof.

You can just refer to the rules by number, i.e. write something like “using (i)” or “by (iv)” etc. A useful starting point might be to consider what happens if $\mathrm{f}(x)$ is a particular constant (such as $\mathrm{f}(x) = 27$, but you will probably find that something other than 27 might be of more use). Using specific examples can often help you get a feel for a question and help suggest what might be useful in a more general case.

You may find it useful to see what happens when you set $\mathrm{g}(x) = \mathrm{f}(x)$.

[4]The precise level of fun will vary between individuals.

Warm down

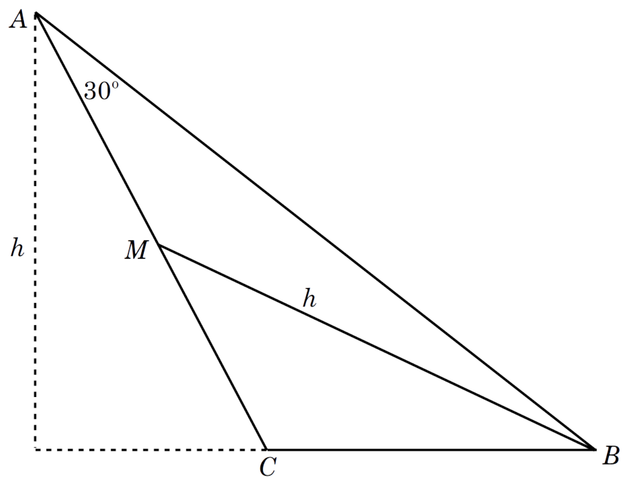
5 The diagram below shows a triangle ABC, with $\angle CAB = 30°$. The point M is the midpoint of AC. The height of the triangle h is the same as the distance from M to B, as shown.

The task is to show that $\angle ABC = 45°$.

You might like to have a go before you try the separate parts below.

There are lots of ways of doing this, but it is possible to get into a terrible mess — especially if you try to use the cosine rule at any stage.

The key is to decide how to use the fact that M is the mid-point of AC.



Here is some blank space to allow you to think how you might do the question without looking at the steps below.

(i) Find the height of triangle MBC in terms of h and show that $\angle MBC = 30°$.

(ii) Show that $\angle BMC = \angle ABC$.

(iii) Find the length of AB in terms of h and $\angle ABC$. Then show that $\sin^2 \angle ABC = \frac{1}{2}$ by means of the sine rule in triangle ABM.