STEP Support Programme

Assignment 21

Warm-up

1 We define two new functions C and S by

$$\mathrm{C}(x) = \tfrac{1}{2}\big(a^x + a^{-x}\big) \quad \text{and} \quad \mathrm{S}(x) = \tfrac{1}{2}\big(a^x - a^{-x}\big)$$

where a is some fixed positive real number.

(i) Show, using these definitions, that

(a) $\big(\mathrm{C}(x)\big)^2 - \big(\mathrm{S}(x)\big)^2 = 1$
(b) $\mathrm{C}(x)\mathrm{C}(y) + \mathrm{S}(x)\mathrm{S}(y) = \mathrm{C}(x+y)$
(c) $\mathrm{C}(x)\mathrm{S}(y) + \mathrm{S}(x)\mathrm{C}(y) = \mathrm{S}(x+y)$.

Deduce an expression for $\mathrm{C}(2x)$ in terms of $\mathrm{C}(x)$.

(ii) *You are given* that the limit as $h \to 0$ of $\dfrac{a^h - 1}{h}$ is K, i.e.

$$\lim_{h\to 0} \frac{a^h - 1}{h} = K\,,$$

where K is some given number[1].

Show from first principles, using the definition of differentiation as given in Assignment 20 question **1(i)**, that

$$\frac{\mathrm{d}(a^x)}{\mathrm{d}x} = Ka^x$$

and that

$$\frac{\mathrm{d}(a^{-x})}{\mathrm{d}x} = -Ka^{-x}$$

Note: $\dfrac{\mathrm{d}(a^x)}{\mathrm{d}x} \neq xa^{x-1}$.

Since a^x does not depend on h, it can be "taken outside" the limit.

(iii) Hence (even if you didn't manage to do part **(ii)** above) find the derivatives of $\mathrm{C}(x)$ and $\mathrm{S}(x)$ and deduce that $\mathrm{C}(x)$ satisfies the differential equation

$$\frac{\mathrm{d}^2\mathrm{C}(x)}{\mathrm{d}x^2} = K^2\,\mathrm{C}(x)\,.$$

You are not asked to *solve* this equation: you just have to substitute $\mathrm{C}(x)$ into the left hand side of the equation and check that it gives the right hand side.

[1] Actually, $K = \ln a$ i.e. $\log_e a$. This can be shown by writing $y = a^x$, taking natural logs of both sides and using implicit differentiation (for those of you who have met these ideas!).

Preparation

You may have already met the modulus function, $|x|$. In case you have not, it is defined as follows:

$$|x| = \begin{cases} +x & \text{for } x \geqslant 0 \\ -x & \text{for } x < 0 \end{cases}$$

The effect of the modulus function is to make the argument of the function (i.e. the input) positive. Removing modulus signs gives either the positive or negative value of the argument. For example, if $|x| = 3$ then either $x = 3$ or $x = -3$.

2 **(i)** Solve the equation $|2x - 3| - 4 = 3$

Start by writing this in the form $|y| = c$. Then you have two equations to solve: $y = c$ and $y = -c$.

(ii) Sketch the graph $y = |2x - 3|$

There are two usual ways to proceed.

One way is to look at the "critical value" of x, where $|2x - 3| = 0$, and split the graph into two regions. In one region $y = 2x - 3$ and in the other region $y = -(2x - 3)$.

The other way is to sketch the graph of $y = 2x - 3$ and then reflect the part that lies under the x-axis about the x-axis (so all the graph now lies above the x-axis). This second method works if the graph is of the form $y = |\mathrm{f}(x)|$, but not for more complicated graphs such as $y = |\mathrm{f}(x)| + |\mathrm{g}(x)|$, or even $y = |\mathrm{f}(x)| + 2$ — though this last one can be found by translating $y = |\mathrm{f}(x)|$.

(iii) Solve the equation $|2x| + |x - 1| = 3$

There are two "critical values" here, at $x = 0$ and $x = 1$.

If $x < 1$ then $|x - 1| = -(x - 1)$ (in order to make it positive), and if $x > 1$ then $|x - 1| = x - 1$. You need to consider three different equations for the three regions separated by the critical values. You must make sure that any solutions you find actually fall in the region you are considering. Substituting your values back into the original equation to check that they work is always a good idea.

(iv) Sketch the graph $|x| + |y| = 1$ in the regions of the x-y plane given by:

- $x > 0$ and $y > 0$;
- $x > 0$ and $y < 0$.

For the first region, $|x| = x$ and $|y| = y$ so the graph is just $x + y = 1$

(v) Sketch the graph $|x - 1| + |y - 1| = 1$ in the region where $x < 1$ and $y > 1$. In this region, shade the subset of the x-y plane in which $|x - 1| + |y - 1| \leqslant 1$.

The STEP question

3 Sketch the following subsets of the x-y plane:

(i) $|x| + |y| \leqslant 1$;

(ii) $|x-1| + |y-1| \leqslant 1$;

(iii) $|x-1| - |y+1| \leqslant 1$;

(iv) $|x|\,|y-2| \leqslant 1$.

It is probably easiest to start each part by sketching the boundaries (given by replacing the inequality signs by equals signs) first. Then all you have to do is test one point in each region.

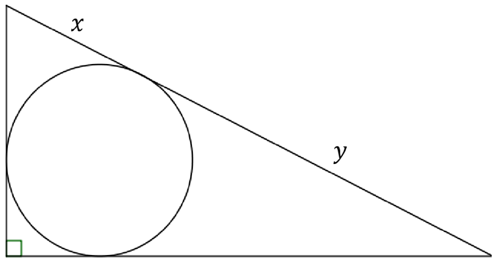
There is a connection between parts **(i)** and **(ii)**. You can use this to help you sketch part **(ii)** and use similar ideas in parts **(iii)** and **(iv)**.

Discussion

In A-Level (or similar) examinations, you might be given a function $\mathrm{f}(x)$ and be asked to sketch $y = |\mathrm{f}(x)|$. This can give interesting-looking graphs but essentially all you have to do is take any parts of the graph that lie underneath the x-axis and reflect them about the x-axis. STEP questions are usually more complicated, and require a deeper understanding of the modulus function. You might like to investigate the differences between $y = |\sin x|$ and $y = \sin |x|$.

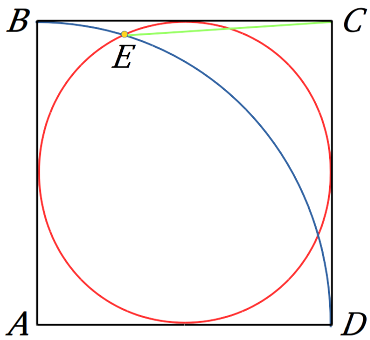
Warm down

4 **(i)** In the diagram below, the circle touches all three sides of the triangle. Find the area of the triangle in terms of x and y.



It will help to draw some radii onto the diagram. It might also be helpful to assign a symbol to represent the radius of the circle.

(ii) In the diagram below, $ABCD$ is a square. The arc BD has centre A and radius AB. It meets the inscribed circle at point E. Show that the length CE is half the length of the diagonal of the square.



There are various ways you can tackle this problem, but the first thing is to decide what the length of AB is going to be. You could make this a or $2a$, or you could argue WLOG[2] that $AB = 2$ (or 1 if you prefer).

It would be a good idea to mark the centre of the square (call it O) on your diagram. You might then like to think about the various triangles with vertices chosen from A, C, O and E. You can then use pure geometry, either by means of similar triangles which is probably the most efficient way or using trigonometry which is a little more labourious.

You can also use coordinate geometry, perhaps by making $A = (0, 0)$, writing down equations for the two circles and finding their points of intersections. There are lots of opportunities to make mistakes with this method!

[2]This means "Without Loss Of Generality"— in this case all squares are congruent, so proving the result for one particular square (with a side length of 2 say) means that it will be true for all squares.