STEP Support Programme

Assignment 22

Warm-up

1 This question is about the product rule for differentiating a product of two functions.

(i) Use a rough sketch to show that (for any function f that can be differentiated)

$$\mathrm{f}(x+h) \approx \mathrm{f}(x) + h\mathrm{f}'(x) \qquad (\dagger)$$

when h is "small".

(ii) The function g is defined by $\mathrm{g}(x) = \mathrm{f}_1(x)\mathrm{f}_2(x)$, where f_1 and f_2 are two given (differentiable) functions. Use the definition $\mathrm{g}'(x) = \lim_{h\to 0} \dfrac{\mathrm{g}(x+h) - \mathrm{g}(x)}{h}$, then $(\dagger)$, to show that

$$\mathrm{g}'(x) = \mathrm{f}_1'(x)\mathrm{f}_2(x) + \mathrm{f}_1(x)\mathrm{f}_2'(x)\,.$$

2 The exponential function e^x is defined by the infinite series:

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad (*)$$

(which converges for all x).

There are various definitions of the function e^x, and you may well know a different definition from the one above. For example, you can define it as the inverse function to the natural logarithm $\ln x$, which is $\log_e x$ (e being a certain number). Or you could define it as simply e^x, where e is the certain number again.[1]

For this question, forget any definitions you know except for $(*)$. You are not required to know this definition for STEP I from 2019 onwards, but it was assumed in older STEP I papers - so you may need it when working through past papers.

(i) Use definition $(*)$ to find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^x\right)$.

(ii) Use definition $(*)$ to find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{kx}\right)$ where k is a constant.

(iii) Let $\mathrm{f}(x) = x\mathrm{e}^x$. Show, using the product rule, that $\mathrm{f}'(x) = (x+1)\mathrm{e}^x$.

Can you get this result from the definition $(*)$ without using the product rule?

[1] In the usual notation (for A-levels, etc), the exponential function is written e^x, using roman type face e to show that it is a function rather than a number, and we follow that convention.

(iv) Use the product rule to show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{ax}\mathrm{e}^{bx}\right) = (a+b)\mathrm{e}^{ax}\mathrm{e}^{bx}\,.$$

Since you are told that e^x is defined by $(*)$, you cannot use any rules of indices here!

Starting with this result, show that $\mathrm{e}^x\mathrm{e}^{-x} = 1$ (which means that you have now shown that $\mathrm{e}^{-x} = \frac{1}{\mathrm{e}^x}$).

Use this result and definition (*) to show that $x\mathrm{e}^x \to 0$ as $x \to -\infty$.

Preparation

3 **(i)** Find the range of values of x for which $3x^2 + x - 2 < 0$.

(ii) Sketch the curve $y = \mathrm{e}^x$.

(iii) This part concerns the curve $y = (x-3)\mathrm{e}^x$.

(a) Differentiate $(x-3)\mathrm{e}^x$ and hence find the coordinates of the stationary point of the curve $y = (x-3)\mathrm{e}^x$. Use the sign of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ to determine the nature of the stationary point.

(b) Find the coordinates of the intersections of the curve with the axes. Determine the values of x for which $(x-3)\mathrm{e}^x$ is negative.

(c) Sketch the curve $y = (x-3)\mathrm{e}^x$. You may assume that $x\mathrm{e}^x \to 0$ as $x \to -\infty$.

(d) Find the values of k for which the equation $(x-3)\mathrm{e}^x = k$ has two roots. Find the values of k for which the equation has one root.

(iv) This part concerns the curve $y = \sin(x^2)$.

(a) Sketch the curve $y = \sin x$ for $-4\pi \leqslant x \leqslant 4\pi$.

(b) Find the first four non-negative values of x for which $\sin(x^2) = 0$.

(c) If $\mathrm{f}(x) = \sin(x^2)$, express $\mathrm{f}(-a)$ in terms of $\mathrm{f}(a)$.

(d) Sketch the curve $y = \sin(x^2)$ for $-4 \leqslant x \leqslant 4$.

The STEP question

4 **(i)** Sketch the curve $y = \mathrm{e}^x(2x^2 - 5x + 2)$.

Hence determine how many real values of x satisfy the equation $\mathrm{e}^x(2x^2 - 5x + 2) = k$ in the different cases that arise according to the value of k.

You may assume that $x^n\mathrm{e}^x \to 0$ *as* $x \to -\infty$ *for any integer* n.

(ii) Sketch the curve $y = \mathrm{e}^{x^2}(2x^4 - 5x^2 + 2)$.

Discussion

When sketching a curve, make sure you consider turning points, intersections with the axes, and the behaviour as $x \to \pm\infty$. You may be sure of the nature of the turning points without having to calculate the second derivative (though you might calculate a second derivative just to be confirm that your sketch is right).

The key to the second part is to work out how the two curves are related.

Warm down

5 Notation: for any polyhedron (i.e. three-dimensional solid whose surface consists of a collection of polygonal faces, joined at their edges), the number of faces is F, the number of edges is E and the number of vertices is V.

(i) Write down F, E and V for a tetrahedron (a pyramid with a triangular base where all the faces are equilateral triangles). Calculate $F - E + V$.

(ii) Repeat part **(i)** for cube.

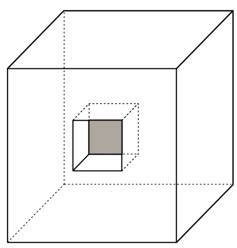
(iii) A regular icosahedron has 20 faces, each of which is an equilateral triangle. 5 faces meet at each vertex of the icosahedron. What is E? What is V? Don't just write down the answers; provide brief justification.

Calculate $F - E + V$ for an icosahedron.

There are 5 different *Platonic Solids*, i.e. solid shapes where each face is a regular polygon. These are:

- A **tetrahedron** (4 equilateral triangle faces, three meet at each vertex)
- A **cube** (6 square faces, three meet at each vertex)
- An **octahedron** (8 equilateral triangle faces, four meet at each vertex)
- A **dodecahedron** (12 regular pentagon faces, three meet at each vertex)
- A **icosahedron** (20 equilateral triangle faces, five meet at each vertex)

(iv) The diagram below shows a cube with a small cubical hole dug into one face. Calculate $F - E + V$.



Discussion

Euler's formula $V - E + F = 2$ holds for *convex* polyhedra (ones where any two points on the surface are connected by a straight line that lies entirely within or on the surface of the polyhedron).

Other three dimensional shapes satisfy $V - E + F = \chi$, where χ is the *Euler characteristic*. For example, a *torus* (ring doughnut shape) had $\chi = 1$. There is lots of information out there on proofs of Euler's formula and different characteristics, so have a search!