STEP Support Programme

Assignment 23

Warm-up

1 This question is all about the *chain rule* for differentiating a composition of functions (a function of a function). If you already know the chain rule, please forget it for the moment.

(i) Let $\mathrm{g}(x) = 2x^3 + 1$ and $\mathrm{f}(x) = x^2$. Let $\mathrm{F}(x) = \mathrm{f}\big(\mathrm{g}(x)\big)$. Find an expression for $\mathrm{F}(x)$ in terms of x. Find $\mathrm{F}'(x)$ and show that $\mathrm{F}'(x) \neq \mathrm{f}'\big(\mathrm{g}(x)\big)$.

(ii) Use the formulae

$$\sin(A+B) = \sin A\cos B + \cos A \sin B \quad \text{and} \quad \cos(A+B) = \cos A \cos B - \sin A \sin B$$

to obtain expressions for $\sin 2\alpha$ and $\cos 2\alpha$ in terms of $\sin\alpha$ and $\cos\alpha$.

Let $\mathrm{f}(x) = \sin x$ and $\mathrm{g}(x) = 2x$. Let $\mathrm{F}(x) = \mathrm{f}\big(\mathrm{g}(x)\big)$. Find an expression for $\mathrm{F}(x)$ in terms of x. Using the rule for differentiating a product, which was derived in Assignment 22, (and without using the chain rule!), show that $\mathrm{F}'(x) = 2\cos(2x)$.

(iii) Now we assume that for any differentiable[1] function H, the approximation

$$\mathrm{H}(a+t) \approx \mathrm{H}(a) + t\mathrm{H}'(a) \qquad (*)$$

is valid when t is small (See Assignment 22 for the explanation of why we can use this).

(a) Use $(*)$ to write down an approximation for $\mathrm{g}(x+h)$, where g is a differentiable function and h is small.

(b) Write down an approximation for $\mathrm{f}\big(\mathrm{g}(x)+t\big)$, where f is a differentiable function and t is small.
Use $(*)$ again, with $\mathrm{H} = \mathrm{f}$.

(c) Let $\mathrm{F}(x) = \mathrm{f}\big(\mathrm{g}(x)\big)$. Use your answers to **(a)** and **(b)** to obtain an approximation for $\mathrm{F}(x+h)$ when h is small.
First step will be to use your approximation for $\mathrm{g}(x+h)$.

(d) Deduce that that $\mathrm{F}'(x) = \mathrm{f}'\big(\mathrm{g}(x)\big)\mathrm{g}'(x)$. This is the *chain rule* for differentiating a 'function of a function', i.e. the composition of two functions.

[1] You don't need to worry about what 'differentiable' means. But to give an example, the function $|x|$ is: continuous everywhere, because you can draw it without taking the pen off the paper; differentiable everywhere except $x = 0$, because its gradient is just ± 1; but not differentiable at $x = 0$ because it is 'pointed' and its gradient can't be calculated.

(iv) Use the chain rule to find the derivative with respect to x of $(2x^3+1)^2$ and $\sin(2x)$.

(v) Differentiate $\ln(2x)$ and $\ln(x^2)$ by two different methods, first using the chain rule and then without using the chain rule. Recall that $\ln x = \log_e x$ and that $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln x = \dfrac{1}{x}$.

Preparation

2 **(i)** Expand $\left(5 - \frac{4}{3}y - 2t\right)^2$.

(ii) Solve the following sets of equations. In each case, sketch the two curves, and indicate the point(s) of intersection if any.

(a) $y = 3x - 1$ and $(x-4)^2 + (y-1)^2 = 9$

(b) $y = 2x - 3$ and $(x-5)^2 + (y-2)^2 = 5$

(c) $3y = x + 5$ and $x^2 + y^2 - 6x - 2y - 15 = 0$

(iii) The straight line L has equation $y = ax - 1$ and the curve C has equation $y = x^2$.

For what range of values of a does L intersect C in two distinct points?

Find the values of a for which L touches C (i.e. meets C without crossing it). Hence write down the equations of the two tangents to C that pass through the point $(0, -1)$.

(iv) Use the formulae for $\sin 2\alpha$ and $\cos 2\alpha$ in terms of $\sin\alpha$ and $\cos\alpha$ to show that

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}\,.$$

The STEP question

3 **(i)** The equation of the circle C is

$$(x-2t)^2+(y-t)^2=t^2,$$

where t is a positive number. Show that C touches the line $y=0$.

Let α be the acute angle between the x-axis and the line joining the origin to the centre of C. Show that $\tan 2\alpha = \frac{4}{3}$ and deduce that C touches the line $3y=4x$.

(ii) Find the equation of the incircle of the triangle formed by the lines $y=0$, $3y=4x$ and $4y+3x=15$.

Note: The *incircle* of a triangle is the circle, lying totally inside the triangle, that touches all three sides.

Discussion

A picture is worth a thousand words.[2] You can do much of part (i) using sketches for which little explanation is needed.

Warning: there is some algebra in part (ii), with big numbers. For the '$b^2=4ac$' condition required for the circle to touch $4y+3x=15$, you can do some cancellation to get $(t-4)^2=(4t^2-20t+25)$.

You then find two possible values of t, and again a sketch would be sufficient to show which is the required value.

[2]This quote is sometimes attributed to Napoleon Bonaparte, who said 'Un bon croquis vaut mieux qu'un long discours' or 'A good sketch is better than a long speech'.

Warm-down

4 Here is the set-up for this question. You should justify your answers as *briefly* as possible.

- We have an island with 100 islanders living on it.
- Islanders have either blue eyes or brown eyes.
- Islanders meet at 9.00 each morning and look into each others' eyes.
- Islanders never discuss eye-colour. There are no mirrors or cameras on the island.
- Any islander who finds out that he or she has blue eyes must leave the island by the 12.00 boat.
- We join the islanders, in spirit anyway, on day one when a ghostly voice at the start of the 9.00 meeting says '*At least one of you has blue eyes*'.

(i) Suppose that one islander leaves the island on the 12.00 boat on day one. What can you conclude?

(ii) Suppose that no islander leaves the island on the 12.00 boat on day one. What can you conclude?

(iii) Suppose instead that no one leaves on the first day, but two islanders leave on the 12.00 boat on the second day. What can you conclude?

(iv) Suppose instead that we don't know if anyone leaves on the first day, but we do know that two islanders leave on the 12.00 boat on the second day. What can you conclude?

(v) Suppose that n islanders leave on the nth day. What can you conclude?

It might be helpful to do part **(ii)** before part **(i)**.