STEP Support Programme

Assignment 24

Warm-up

1 **(i)** *Integration as inverse of differentiation*[1]

The Fundamental Theorem of Calculus says[2]:

$$\frac{\mathrm{dF}}{\mathrm{d}x} = \mathrm{f}(x) \iff \mathrm{F}(x) - \mathrm{F}(a) = \int_a^x \mathrm{f}(t)\,\mathrm{d}t\,,$$

or, if you prefer,

$$\frac{\mathrm{dF}}{\mathrm{d}x} = \mathrm{f}(x) \iff \mathrm{F}(x) = \int \mathrm{f}(x)\,\mathrm{d}x + C\,,$$

where C is a constant. If you are not sure of this, try it with $\mathrm{f}(x) = x^3$ and $\mathrm{F}(x) = \frac{1}{4}x^4$.

(a) In Assignment 22, you showed that $\dfrac{\mathrm{d\,e}^{kx}}{\mathrm{d}x} = k\mathrm{e}^{kx}$ (where k is a constant). Use this result to integrate

$$\int_0^{\ln 2} \mathrm{e}^{4t}\mathrm{d}t\,.$$

(b) Use the chain rule to differentiate $\sin(kx)$ and $\cos(kx)$ (where k is a constant). Integrate

$$\int_0^{\frac{1}{2}\pi} \sin(2t)\,\mathrm{d}t\,.$$

(c) Use the product rule to differentiate $x \sin x$. Evaluate

$$\int_0^{\pi} (x\cos x + \sin x)\,\mathrm{d}x\,.$$

Find

$$\int (x\sin x - \cos x)\,\mathrm{d}x\,.$$

This sort of method is sometimes called "Integration by inspection", which basically means have a guess at what the answer might be and then differentiate to check.

[1] Sometimes horribly called 'anti-differentiation'.

[2] Note the use of a different letter for the variable in the first integral. Here t is called a 'dummy variable': it doesn't matter what letter you use because it will disappear when you do the integral and evaluate the result at $t = x$ and $t = a$. The only letters you should not use are x and a, because they have been used as limits.

(ii) *Integration by parts*

In Assignment 22, you proved the product rule:

$$\frac{\mathrm{d}(uv)}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}\,,$$

i.e.

$$u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}(uv)}{\mathrm{d}x} - v\frac{\mathrm{d}u}{\mathrm{d}x}\,.$$

Integrating both sides of this last equation gives

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x\,,$$

which is the formula for integrating by parts.

(a) Integrate $x\mathrm{e}^x$.

(b) Evaluate $\displaystyle\int_0^{\frac{1}{2}\pi} x\sin x\,\mathrm{d}x$.

(c) By writing $\ln x = 1 \times \ln x$ and remembering that $\dfrac{\mathrm{d}\ln x}{\mathrm{d}x} = \dfrac{1}{x}$, integrate $\ln x$.

(d) Let $I = \displaystyle\int \mathrm{e}^x \sin x\,\mathrm{d}x$. By using integration by parts twice, show that:

$$I = \mathrm{e}^x \sin x - \mathrm{e}^x \cos x - I$$

and hence find I. Don't forget the constant of integration, but you don't need to include it until the final stage.

Differentiate your I to check your answer.

Preparation

2 **(i)** Simplify $\sin(A+B) - \sin(A-B)$.

(ii) The function cosec is defined, for $\theta \neq n\pi$, by $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$. Find the values of:

(a) $\operatorname{cosec} \frac{1}{4}\pi$;

(b) $\operatorname{cosec} \frac{5}{6}\pi$.

(iii) Simplify:

$$\sum_{k=1}^{n} \left(\sqrt{k} - \sqrt{k-1}\right) .$$

(iv) Write down the values of $\cos\pi$, $\cos 2\pi$ and $\cos 3\pi$, and give an expression for $\cos n\pi$.

(v) Integrate $\cos(kx)$ with respect to x.

(vi) Let $I = \displaystyle\int (x\cos x + \sin x)\,\mathrm{d}x$ and $J = \displaystyle\int \sin x\,\mathrm{d}x$.

By considering $I - J$, and using your answer to question **1** **(i)(c)**, find $\displaystyle\int x\cos x\,\mathrm{d}x$.

Yes, you could do this by integration by parts! However you need to use the method dictated — if you are ever given an instruction (such as “hence”) you must follow it.

The STEP question

3 The integral I_n is defined by

$$I_n = \int_0^{\pi} \left(\tfrac{1}{2}\pi - x\right) \sin(nx + \tfrac{1}{2}x) \operatorname{cosec}(\tfrac{1}{2}x) \, \mathrm{d}x \,,$$

where n is a positive integer.

Evaluate $I_n - I_{n-1}$, and hence evaluate I_n leaving your answer in the form of a sum.

Discussion

The notation I_n looks a bit frightening at first, but the subscript n is needed because we want to consider the integral for different values of the constant n in the integrand. The expression for I_n in terms of I_{n-1} is called a *recurrence relation*, which provides a useful method of evaluating such integrals.

You might worry about what happens to the integrand when $x = 0$, when $\sin \frac{1}{2}x$ in the denominator (from the cosec term) is zero. In fact this not a problem, because the sine in the numerator is also zero at $x = 0$ in such a way that zero divided by zero is quantifiable. In this case, we can approximate both sine functions using $\sin\theta \approx \theta$, which is valid for small θ, giving

$$\frac{\sin(nx + \frac{1}{2}x)}{\sin(\frac{1}{2}x)} \approx \frac{(n + \frac{1}{2})x}{\frac{1}{2}x} = 2n + 1 \,.$$

Thus the integrand is well behaved at $x = 0$[3].

On reflection, this is quite short for a STEP question. The main difficulty arises because you are left to your own devices for much of the question.

When you have done the warm down problem, you will see that $I_n \approx \frac{1}{2}\pi^2$, for very large n.

[3]An example of a function which is not "well behaved" when $x = 0$ is $\sin \frac{1}{x}$. Try using Desmos to plot it.

Warm down

4 A famous problem of the early 18th century, called the *Basel problem*, was to evaluate the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This problem had baffled many great mathematicians, in particular those of the Bernoulli family who lived in Basel, Switzerland (whence the name attached to the problem). But in the period 1735–1741, or thereabouts, Euler provided no fewer than five different ways of evaluating the sum.

Several ways led to the result

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \tfrac{1}{8}\pi^2 \,, \qquad (*)$$

(i.e. the sum of the odd terms) from which Euler was able to deduce full result.

(i) Let $S = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ and let $S_{\text{even}} = \sum_{n=1}^{\infty} \dfrac{1}{(2n)^2}$. Write down a relation between S and S_{even}.

Hence use Euler's result $(*)$ to evaluate S.

(ii) Evaluate $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2}$ and $\sum_{n=1}^{\infty} \dfrac{\cos \frac{1}{2}n\pi}{n^2}$.

(iii) Evaluate $\sum_{n=1}^{\infty} \left(\dfrac{1}{(6n-5)^2} + \dfrac{1}{(6n-1)^2} \right)$.