STEP Support Programme

Assignment 3

Warm-up

1 **(i)** Evaluate $\sum_{n=0}^{6} \sin(\frac{n\pi}{6})$.

Here, $\frac{n\pi}{6}$ is in radians, and 'evaluate' means give the exact value — so leave it in the form of surds. For those of you have not met the "$\sum$" (sigma) notation before $\sum_{n=0}^{6} \sin(\frac{n\pi}{6})$ means $\sin(\frac{0\times\pi}{6}) + \sin(\frac{1\times\pi}{6}) + ... + \sin(\frac{6\times\pi}{6})$, i.e you substitute the values of n from 0 to 6 inclusive and sum the results.

(ii) Let $S_n = \sum_{i=0}^{n-1} r^i$ (which is n terms altogether). Simplify $rS_n - S_n$ and hence find a formula, in the case $r \neq 1$, for S_n. What is the corresponding result when $r = 1$?

Deduce a formula for $\sum_{i=0}^{n-1} ar^i$.

If we know that $-1 < r < 1$, what can be said about r^n as n gets very large? Deduce a formula for $\sum_{i=0}^{\infty} ar^i$ given that $-1 < r < 1$.

If you are not very used to working with the $\sum$ notation, you may prefer to write out the sums with dots (for example $1 + r + r^2 + \cdots + r^{n-1}$). It is conventional to use exactly three dots.

(iii) Evaluate $\sum_{i=0}^{9} 3\left(\frac{1}{2}\right)^i$.

Preparation

2 **(i)** Sketch the graph of $y = 2x + 1$. Use your graph and the formula for the area of a trapezium (or some similar formulae if you prefer) to evaluate

$$\int_2^5 (2x+1)\,dx$$

i.e. find the area under the graph of $y = 2x + 1$ between $x = 2$ and $x = 5$.

You may want to check your answer by "doing" the integral. However the preparation and STEP question require only the idea that integral means the area under a curve — you don't need any other techniques for evaluating integrals.

(ii) The square bracket ('floor')[1] notation $[x]$ means the greatest integer less than or equal to x. For example, $[\pi] = 3$, $[\sqrt{24}\,] = 4$ and $[5] = 5$.

What are the integer values of $[10.2]$, $[\sqrt{70}]$, $[6]$ and $[10\pi]$?

(iii) If $3 \leqslant x < 4$, what is $[x]$? Sketch the graph of $y = [x]$ for $0 \leqslant x < 4$.

Note that this graph is discontinuous — it should be a collection of separate parts. Do not join the bits with vertical solid lines (dotted would be ok).

(iv) Use your sketch in part **(iii)** to evaluate

$$\int_0^4 [x]\,dx\,.$$

(v) Sketch the graph of $y = x[x]$ for $0 \leqslant x < 3$. Use your sketch to evaluate

$$\int_0^3 x[x]\,dx\,.$$

Consider what the function will be for different ranges of x, i.e. $0 \leqslant x < 1$ etc. This graph is again a collection of disconnected straight line segments.

[1] The square bracket $[a]$ and the floor bracket $\lfloor a \rfloor$ are equivalent notations for 'integer part'. Carl Friedrich Gauss introduced the square bracket notation in his third proof of quadratic reciprocity (1808). This remained the standard notation in mathematics until Kenneth E. Iverson introduced the names "floor" and "ceiling" and the corresponding notations in his 1962 book A Programming Language.

The STEP question

3 The square bracket notation $[x]$ means the greatest integer less than or equal to x. For example, $[\pi] = 3$, $[\sqrt{24}\,] = 4$ and $[5] = 5$.

(i) Sketch the graph of $y = \sqrt{[x]}$ and show that

$$\int_0^a \sqrt{[x]}\,\mathrm{d}x = \sum_{r=0}^{a-1} \sqrt{r}$$

when a is a positive integer.

(ii) Show that $\displaystyle\int_0^a 2^{[x]}\,\mathrm{d}x = 2^a - 1$ when a is a positive integer.

(iii) Determine an expression for $\displaystyle\int_0^a 2^{[x]}\,\mathrm{d}x$ when a is positive but not an integer.

Discussion

This is very typical of STEP: the question involves a function (or process or notation) that is — or may be — unfamiliar to you but which is defined carefully in the question; then you are asked to use the function for progressively more difficult tasks. You should not be put off by this. It may take a little time to understand the definition, but then you should be able to apply it with confidence.

It is also typical in that there are quite a few different concepts in a single question: integer part, integration (as area), summation notation and geometric progression.

As is often the case, clear diagrams are invaluable. The last part is rather clever: once you have got into the question, you will find that you need to use the floor function again.

Warm down

4 Arthur, Brenda, and Chandrima, together with an orangutan, are shipwrecked on a desert island. They spend the first day gathering a total of N bananas which they heap up together into one pile. Then they go to sleep for the night.

In the middle of the night, Arthur wakes up. He decides to take his share of the bananas immediately in case there is an argument about it in the morning. He divides the bananas into three equal piles but finds that there is one banana left over. This he gives to the orangutan. He then hides his pile, heaps the rest together and goes back to sleep.

Shortly afterwards, Brenda wakes up and does the same thing, dividing the pile that she finds into three equal piles and hiding her pile. She also has one left over, which she gives to the orangutan before going back to sleep.

Shortly after this, Chandrima wakes up and does the same thing, taking one third of the bananas in the pile that she finds, after giving one to the orangutan.

In the morning the three of them divide the remaining bananas, which comes out in three equal shares of m bananas, with one left over for the orangutan. Of course each one knows that there are bananas missing; but each one is as guilty as the others, so no one says anything.

(i) Show that

$$8N = 81m + 65 \qquad (*)$$

It will be helpful to note that the number of bananas left after Chandrima has taken her share is $3m + 1$.

(ii) Show that if N satisfies $(*)$ for some integer m, then $N + 81$ also satisfies $(*)$ (for a different value of m).

If N satisfies $(*)$ this means that there exists an integer m such that $8N = 81m + 65$. You then have to consider $N' = N + 81$ and show that there is an integer m' such that $8N' = 81m' + 65$.

(iii) Show that $N = -2$ (two negative bananas) is a solution of $(*)$ (i.e. there exists an integer m such that $8N = 81m+65$). Find a solution for a positive number of bananas.

If you have time, you might like to find a general formula for the integers N and m that can satisfy $(*)$ (you would need to show that there are no others).