STEP Support Programme

Assignment 5

Warm-up

1 **(i)** The triangle ABC has a right-angle at C. The lengths of the sides BC, CA and AB are a, b and c, respectively. Angle CAB is θ. Express $\cos\theta$ and $\sin\theta$ in terms of a, b and c. Hence show that $\cos^2\theta + \sin^2\theta = 1$.

(ii) The points A, B and C have coordinates (x, y), $(a, 0)$ and $(0, 0)$, respectively, where a, x and y are all positive. The lengths AB and AC are c and b, respectively. Write down expressions for b^2 and c^2 in terms of x, y and a. Hence show that

$$b^2 - x^2 = c^2 - (x-a)^2\,.$$

Let $\angle ACB = C$. Express x in terms of b and C. Deduce that

$$c^2 = a^2 + b^2 - 2ab\cos C\,.$$

Is there a difference between the cases $a > x$ and $a < x$?

Preparation

2 **(i)** *In this question, you should leave your answers as fractions in their lowest terms.*

The triangle ABC has $AB = 10$, $BC = 9$ and $CA = 17$.

Find the value of $\cos C$.

Find the value of $\sin C$ (using Question **1(i)**). Hence show that the area of the triangle is 36.

Use the area of the triangle to find the lengths of three altitudes (note: an altitude is a perpendicular from one side to the opposite vertex, so that the length of the altitude is the height of the triangle when it is standing on that particular side).

(ii) Three of the vertices of the base of a rectangular-based pyramid are at $(0, 0, 0)$, $(4, 0, 0)$ and $(0, 6, 0)$. Find the coordinates of the fourth vertex of the base.

Given that the volume of the pyramid is 40, find the height. (Note: the volume of a pyramid is $\frac{1}{3} \times$ area of base $\times$ height.)

Given that the apex of the pyramid is directly over the centre of the base, write down its coordinates.

(iii) Simplify, in the case $x \geqslant 0$,

$$\left(1 - \frac{1}{1+x^2}\right)^{\frac{1}{2}} \sqrt{1+x^2}\ .$$

What is the answer if $x < 0$?

The STEP question

3 *Note that the volume of a tetrahedron is equal to* $\frac{1}{3} \times$ *the area of the base* $\times$ *the height.*

The points O, A, B and C have coordinates $(0,0,0)$, $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$, respectively, where a, b and c are positive.

(i) Find, in terms of a, b and c, the volume of the tetrahedron $OABC$.

(ii) Let angle $ACB = \theta$. Show that

$$\cos\theta = \frac{c^2}{\sqrt{(a^2+c^2)(b^2+c^2)}}$$

and find, in terms of a, b and c, the area of triangle ABC.

Hence show that d, the perpendicular distance of the origin from the triangle ABC, satisfies

$$\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\,.$$

Discussion

Note that the preparation questions are meant to be very relevant and helpful! (As are the warm up questions in this case).

It will be helpful to draw a good diagram. When using the cosine rule you need to be careful: the length AB is **not** equal to c. A more helpful version for this question is $AB^2 = AC^2 + BC^2 - 2 \times AC \times BC \times \cos\theta$.

For the last part, the algebra is a bit intimidating. It should be OK if you hold your nerve. Remember that you know the volume of the tetrahedron.

Warm down

I have a drawer full of identical red socks and identical blue socks.

4 **(i)** How many socks must I take from the drawer to be sure that I have a matching pair?

(ii) How many socks must I take from the drawer to be sure that I have 2 matching pairs (not necessarily four socks of the same colour)?

(iii) How many socks must I take from the drawer to be sure that I have n matching pairs?

Don't just write down the answer: you should explain your reasoning. The first two parts can be solved by listing possible outcomes, and these answers should give you an idea of what the general result is. However *extrapolation* is not a *proof* and you need to justify the general case fully.