STEP Support Programme

Assignment 6

Warm-up

1 **(i)** Find the value of

$$\frac{(1+\frac{1}{2})(1+\frac{1}{4})(1+\frac{1}{6})(1+\frac{1}{8})}{(1-\frac{1}{2})(1-\frac{1}{4})(1-\frac{1}{6})(1-\frac{1}{8})}\,.$$

Find the value (in terms of n) of

$$\frac{(1+\frac{1}{2})(1+\frac{1}{4})(1+\frac{1}{6})(1+\frac{1}{8})\cdots(1+\frac{1}{2n})}{(1-\frac{1}{2})(1-\frac{1}{4})(1-\frac{1}{6})(1-\frac{1}{8})\cdots(1-\frac{1}{2n})}\,.$$

(ii) Solve the simultaneous equations:

$$\begin{aligned} a+b-c&=2\\ a-b+c&=0\\ -a+b+c&=8\,. \end{aligned}$$

Now solve the simultaneous equations:

$$\begin{aligned} ka+b-c&=2\\ a-b+c&=0\\ -a+b+c&=8\,. \end{aligned}$$

where k is a fixed but unknown number. Are there any values of k for which the equations have no solution?

Preparation

2 This preparation is all about arrangements of letters. For those of you who have not met arrangements before there are examples at the end of this assignment.

(i) In how many ways can Ben, Elsa and Charlie arrange the letters in their names?

(ii) In how many distinct ways can Emma, David and Stephen arrange the letters of their names?

In how many distinct ways can Poppy arrange the letters of her name?

In how many distinct ways can Ebenezer arrange the letters of his name?

(iii) How many distinct ways are there of arranging the letters in Reggie, Hannah and Marshmallow?

(iv) Lillian now wants a go. Show carefully that the number of distinct ways she can write her name is $\frac{7!}{3!\times 2!}$.

(v) In how many distinct ways can you arrange the letters in MISSISSIPPI?

The STEP question (2005 STEP I Q1)

3 47231 is a five-digit number whose digits sum to $4+7+2+3+1=17$.

(i) Show that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning clearly.

(ii) How many five-digit numbers are there whose digits sum to 39?

Warm down

4 **(i)** A study of a large population found that 40% were men and 60% were women. Of the men 50% were smokers, and of the women 30% were smokers.

You may wish to refer to one or other of the diagrams below (one is a tree diagram and one is similar to a Venn diagram) to help you; or you may want to model the situation by considering a population of 100 people.

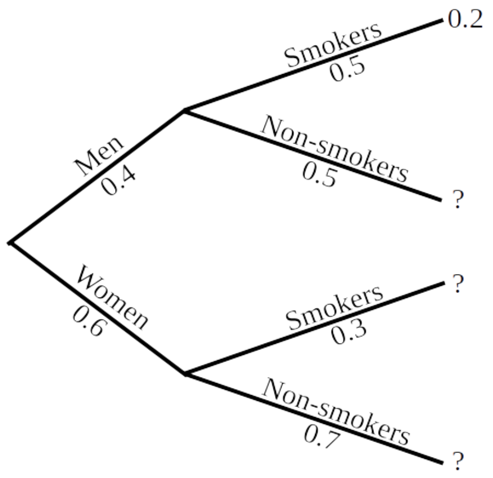
(a) What is the probability that a person picked at random[1] is a female smoker?

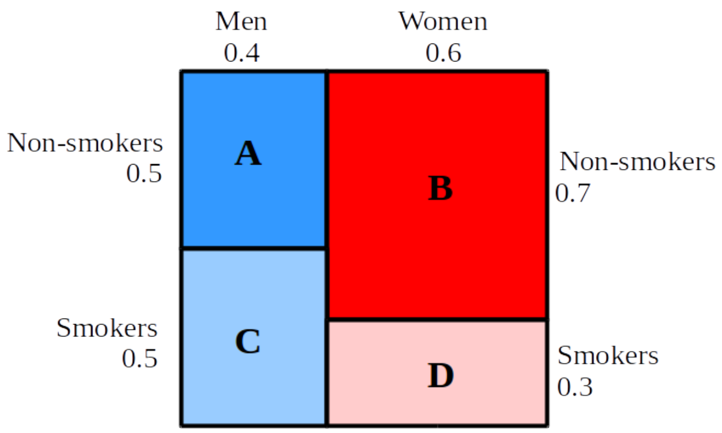
(b) What is the probability that a person picked at random is a non-smoker?

(c) Given that the person picked is a woman, what is the probability that she is a smoker?

(d) Given that the person picked is a smoker, show that the probability that he or she is a woman is $9/19$.

(e) Given that the person picked is a non-smoker, find the probability that he or she is a man.





[1]Here *random* means that any person has the same probability of being picked as any other person.

(ii) The disease "Mathmotitus" affects 0.1% of the population. There is a blood test that gives the correct result in 99% of people who do have the disease and 98% of people who do not have the disease. You have just tested positive, how worried should you be (i.e. what is the probability that you have the disease)?

Discussion

Blood tests are never 100% accurate. There are two ways in which the test can be incorrect. The first is that it can give a *false-negative* result, meaning that a person with the disease gets a negative result. This can obviously can have bad consequences for the patient. The second way a test can be incorrect is when is gives a *false-positive* result. This is when a patient without the disease gets a positive result. This also has unfortunate consequences — the patient may well suffer stress, and may be subjected to more invasive diagnostic tests.

The *sensitivity* of a test is the test's ability to identify correctly patients who have the disease. In our example, the sensitivity was 99% meaning that 99% of patients who test positive do in fact have it. The *specificity* of a test is the test's ability to correctly identify patients who do not have the disease, so in our example it is 98%. There is often a trade off between sensitivity and specificity: increasing the sensitivity of a test can decrease the specificity.

Another thing to notice is that P(test positive | have the disease) — i.e. the probability that you test positive if you have the disease (in our example this is 99%) — is very different to P(have the disease | test positive) — i.e. the probability that you have the disease if you have tested positive — but the second probability is of far more importance to you!

Arrangements Examples

5 **(i)** Claire wants to work out the number of ways in which she can arrange the letters of her name (which are all different). There are 6 different positions in which she can place the first letter. For **each** of these positions, there are now 5 places she can place the second letter. Now there are 4 places in which to put the third letter, etc.

The total number of ways of arranging the letters in Claire's name is therefore

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720\,.$$

We can write this as $6! = 720$ where $6!$ is read as "6 factorial".

(ii) Stuart wants work out the number of different (or 'distinct') ways in which he can arrange the letters of his name. As he has a repeated letter in his name he has to be careful (because not all arrangements are distinct). He decides to deal with the repeated T's last.

There are 6 places to put the S, then 5 places for the U, 4 places for the A and 3 places for the R. The T's then take the remaining two places (and there is only 1 option for them). The total number of distinct arrangements is therefore $6 \times 5 \times 4 \times 3 = 360$. We can write this number as $\dfrac{6!}{2!}$.

(iii) Anna wants to join in. As she has two different letters that repeat she tries something slightly different. She starts by treating the letters as if they are all distinct, and names them A_1, N_1, N_2, A_2. There are $4! = 24$ ways of arranging these as before. But for every arrangement there is another arrangement in which the positions of A_1 and A_2 are swapped; and these two arrangements are identical (i.e. not distinct) if A_1 and A_2 are the same. She therefore divides the number of arrangements (i.e. $4!$) by 2 to take account of this duplication. Then she divides by 2 again to take account of the duplication when N_1 and N_2 are the same. She finds that there are $\frac{4!}{2\times 2}$ distinct ways of arranging the letters in her name. This might more usefully be written as $\frac{4!}{2!\times 2!}$.