

## STEP Support Programme

### STEP 3 Algebra: Hints

- 1** For the base case, substitute  $n = 1$ . The given definition for  $f(x)$  makes this true, so not a lot of work needed here.

For the induction you will need to use  $(*)$  twice, once with  $n = k$  and once with  $n = k + 1$ .

If  $f(x) \equiv 0$  then we have  $(T_n(x))^2 = T_{n-1}(x)T_{n+1}(x)$  which can be rearranged to give something useful.

Once you have an expression for  $T_n(x)$  in terms of  $r(x)$  and  $T_0(x)$  you can substitute this into  $(*)$ . You should end up with a quadratic in  $r(x)$ .

- 2** To show the given identity it is probably easier to start on the RHS.

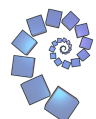
Multiplying both sides of the equation by  $2\sin\frac{1}{2}\theta$  will be helpful, but be aware that this might introduce “extra solutions”.

You can find a useful formula  $\sin A - \sin B$  by setting  $A = \frac{1}{2}(A + B) + \frac{1}{2}(A - B)$  and  $B = \frac{1}{2}(A + B) - \frac{1}{2}(A - B)$ . You might like to refer to the “general solutions” part of the topic notes for STEP 2 Trigonometry.

- 3** (i) Start by simplifying the first two brackets, then this result and the third bracket etc. until you spot the pattern. Setting  $(1 - x)(1 + x)(1 + x^2) \dots (1 + x^{2^n})$  equal to the simplified result and rearranging will lead to the first deduction. You will need to consider what happens to the last term as  $n \rightarrow \infty$ .

Remember that  $\ln a^{-1} = -\ln a$  and  $\ln(A \times B) = \ln A + \ln B$ .

- (ii) You can do this part in the same way as part (i). Use the result here and the final result from part (i) to find some brackets to multiply together which might be useful (like the bit you are given to simplify at the start of part (i)).



- 4 (i) If  $\alpha$  is a root of  $x^2 + ax + b = 0$  then  $\alpha^2 + a\alpha + b = 0$ . Find another equation and eliminate  $\alpha^2$ . Since you are given  $a \neq c$  you can divide by  $a - c$ .

For an “if and only if” you need to show that the argument runs both ways, i.e. you need to prove that:

$$(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0 \implies \text{the equations have at least one common root}$$

**and**

$$\text{the equations have at least one common root} \implies (b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0.$$

You may find one direction easier than the other. Remember that you have:

$$\text{the equations have at least one common root} \implies \alpha = -\frac{b-d}{a-c}.$$

For the last part, you need to show whether the result holds if  $a = c$ .

- (ii) Again this is an “if and only if”.

If we have  $(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0$  then you can use part (i) to write down two quadratic equations which have at least one common root. Try to manipulate these to get a (relevant) cubic.

If the two given equations have at least one common root ( $\alpha$  say), then you can write down two equations involving  $\alpha$ . Eliminate  $\alpha^3$  to end up with two quadratics.

