STEP Support Programme

STEP 3 Algebra: Hints

1 For the base case, substitute $n = 1$. The given definition for $\mathrm{f}(x)$ makes this true, so not a lot of work needed here.

For the induction you will need to use $(*)$ twice, once with $n = k$ and once with $n = k + 1$.

If $\mathrm{f}(x) \equiv 0$ then we have $\Big(\mathrm{T}_n(x)\Big)^2 = \mathrm{T}_{n-1}(x)\mathrm{T}_{n+1}(x)$ which can be rearranged to give something useful.

Once you have an expression for $\mathrm{T}_n(x)$ in terms of $\mathrm{r}(x)$ and $\mathrm{T}_0(x)$ you can substitute this into $(*)$. You should end up with a quadratic in $\mathrm{r}(x)$.

2 To show the given identity it is probably easier to start on the RHS.

Multiplying both sides of the equation by $2\sin\frac{1}{2}\theta$ will be helpful, but be aware that this might introduce "extra solutions".

You can find a useful formula $\sin A - \sin B$ by setting $A = \frac{1}{2}(A + B) + \frac{1}{2}(A - B)$ and $B = \frac{1}{2}(A + B) - \frac{1}{2}(A - B)$. You might like to refer to the "general solutions" part of the topic notes for STEP 2 Trigonometry.

3 **(i)** Start by simplifying the first two brackets, then this result and the third bracket etc. until you spot the pattern. Setting $(1 - x)(1 + x)\left(1 + x^2\right)\dots\left(1 + x^{2^n}\right)$ equal to the simplified result and rearranging will lead to the first deduction. You will need to consider what happens to the last term as $n \to \infty$.

Remember that $\ln a^{-1} = -\ln a$ and $\ln\left(A \times B\right) = \ln A + \ln B$.

(ii) You can do this part in the same way as part **(i)**. Use the result here and the final result from part **(i)** to find some brackets to multiply together which might be useful (like the bit you are given to simplify at the start of part **(i)**).

4 **(i)** If α is a root of $x^2 + ax + b = 0$ then $\alpha^2 + a\alpha + b = 0$. Find another equation and eliminate α^2. Since you are give $a \neq c$ you can divide by $a - c$.

For an "if and only if" you need to show that the argument runs both ways, i.e. you need to prove that:

$(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0 \implies$ the equations have at least one common root

and

the equations have at least one common root $\implies (b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0\,.$

You may find one direction easier than the other. Remember that you have:

$$\text{the equations have at least one common root} \quad \implies \quad \alpha = -\frac{b-d}{a-c}\,.$$

For the last part, you need to show whether the result holds if $a = c$.

(ii) Again this is an "if and only if".

If we have $(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0$ then you can use part **(i)** to write down two quadratic equations which have at least one common root. Try to manipulate these to get a (relevant) cubic.

If the two given equations have a least one common root (α say), then you can write down two equations involving α. Eliminate α^3 to end up with two quadratics.