STEP Support Programme

STEP 3 Algebra Questions

1 **2008 S3 Q5**

The functions $\mathrm{T}_n(x)$, for $n = 0, 1, 2, \ldots$, satisfy the recurrence relation

$$\mathrm{T}_{n+1}(x) - 2x\mathrm{T}_n(x) + \mathrm{T}_{n-1}(x) = 0 \qquad (n \geqslant 1). \qquad (*)$$

Show by induction that

$$(\mathrm{T}_n(x))^2 - \mathrm{T}_{n-1}(x)\mathrm{T}_{n+1}(x) = \mathrm{f}(x)\,,$$

where $\mathrm{f}(x) = (\mathrm{T}_1(x))^2 - \mathrm{T}_0(x)\mathrm{T}_2(x)$.

In the case $\mathrm{f}(x) \equiv 0$, determine (with proof) an expression for $\mathrm{T}_n(x)$ in terms of $\mathrm{T}_0(x)$ (assumed to be non-zero) and $\mathrm{r}(x)$, where $\mathrm{r}(x) = \mathrm{T}_1(x)/\mathrm{T}_0(x)$. Find the two possible expressions for $\mathrm{r}(x)$ in terms of x.

2 **2003 S3 Q6**

Show that

$$2\sin\tfrac{1}{2}\theta\,\cos r\theta = \sin\left(r+\tfrac{1}{2}\right)\theta - \sin\left(r-\tfrac{1}{2}\right)\theta\,.$$

Hence, or otherwise, find all solutions of the equation

$$\cos a\theta + \cos(a+1)\theta + \cdots + \cos(b-2)\theta + \cos(b-1)\theta = 0\,,$$

where a and b are positive integers with $a < b-1$.

3 **2012 S3 Q2**

In this question, $|x| < 1$ and you may ignore issues of convergence.

(i) Simplify
$$(1-x)(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n})\,,$$
where n is a positive integer, and deduce that
$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) + \frac{x^{2^{n+1}}}{1-x}\,.$$
Deduce further that
$$\ln(1-x) = -\sum_{r=0}^{\infty} \ln\left(1+x^{2^r}\right)\,,$$
and hence that
$$\frac{1}{1-x} = \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \cdots\,.$$

(ii) Show that
$$\frac{1+2x}{1+x+x^2} = \frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \cdots\,.$$

4 **2010 S3 Q4**

(i) The number α is a common root of the equations $x^2+ax+b=0$ and $x^2+cx+d=0$ (that is, α satisfies both equations). Given that $a \neq c$, show that
$$\alpha = -\frac{b-d}{a-c}\,.$$
Hence, or otherwise, show that the equations have at least one common root if and only if
$$(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0\,.$$
Does this result still hold if the condition $a \neq c$ is not imposed?

(ii) Show that the equations $x^2+ax+b=0$ and $x^3+(a+1)x^2+qx+r=0$ have at least one common root if and only if
$$(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0\,.$$
Hence, or otherwise, find the values of b for which the equations $2x^2+5x+2b=0$ and $2x^3+7x^2+5x+1=0$ have at least one common root.