

STEP Support Programme

STEP 3 Algebra Questions

1 2008 S3 Q5

The functions $T_n(x)$, for $n = 0, 1, 2, \dots$, satisfy the recurrence relation

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0 \quad (n \geq 1). \quad (*)$$

Show by induction that

$$(T_n(x))^2 - T_{n-1}(x)T_{n+1}(x) = f(x),$$

where $f(x) = (T_1(x))^2 - T_0(x)T_2(x)$.

In the case $f(x) \equiv 0$, determine (with proof) an expression for $T_n(x)$ in terms of $T_0(x)$ (assumed to be non-zero) and $r(x)$, where $r(x) = T_1(x)/T_0(x)$. Find the two possible expressions for $r(x)$ in terms of x .

2 2003 S3 Q6

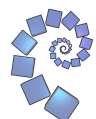
Show that

$$2 \sin \frac{1}{2}\theta \cos r\theta = \sin \left(r + \frac{1}{2}\right)\theta - \sin \left(r - \frac{1}{2}\right)\theta.$$

Hence, or otherwise, find all solutions of the equation

$$\cos a\theta + \cos(a+1)\theta + \dots + \cos(b-2)\theta + \cos(b-1)\theta = 0,$$

where a and b are positive integers with $a < b - 1$.



3 2012 S3 Q2

In this question, $|x| < 1$ and you may ignore issues of convergence.

(i) Simplify

$$(1-x)(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}),$$

where n is a positive integer, and deduce that

$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) + \frac{x^{2^{n+1}}}{1-x}.$$

Deduce further that

$$\ln(1-x) = -\sum_{r=0}^{\infty} \ln(1+x^{2^r}),$$

and hence that

$$\frac{1}{1-x} = \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \cdots.$$

(ii) Show that

$$\frac{1+2x}{1+x+x^2} = \frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \cdots.$$

4 2010 S3 Q4

(i) The number α is a common root of the equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ (that is, α satisfies both equations). Given that $a \neq c$, show that

$$\alpha = -\frac{b-d}{a-c}.$$

Hence, or otherwise, show that the equations have at least one common root if and only if

$$(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0.$$

Does this result still hold if the condition $a \neq c$ is not imposed?

(ii) Show that the equations $x^2 + ax + b = 0$ and $x^3 + (a+1)x^2 + qx + r = 0$ have at least one common root if and only if

$$(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0.$$

Hence, or otherwise, find the values of b for which the equations $2x^2 + 5x + 2b = 0$ and $2x^3 + 7x^2 + 5x + 1 = 0$ have at least one common root.

