

STEP Support Programme

STEP 3 Algebra: Solutions

1 Base Case Letting $n = 1$ we have:

$$\begin{aligned} (\mathbb{T}_n(x))^2 - \mathbb{T}_{n-1}(x)\mathbb{T}_{n+1}(x) &= f(x) \\ (\mathbb{T}_1(x))^2 - \mathbb{T}_0(x)\mathbb{T}_2(x) &= f(x) \end{aligned}$$

Which is true for the given $f(x)$, hence the result is true when $n = 1$.

Inductive Step Assume the result is true when $n = k$, i.e. we have:

$$(\mathbb{T}_k(x))^2 - \mathbb{T}_{k-1}(x)\mathbb{T}_{k+1}(x) = f(x)$$

Now consider the LHS when $n = k + 1$.

$$\begin{aligned} (\mathbb{T}_{k+1}(x))^2 - \mathbb{T}_k(x)\mathbb{T}_{k+2}(x) &= \mathbb{T}_{k+1}(x)(2x\mathbb{T}_k(x) - \mathbb{T}_{k-1}(x)) - \mathbb{T}_k(x)(2x\mathbb{T}_{k+1}(x) - \mathbb{T}_k(x)) \\ &= \cancel{2x\mathbb{T}_{k+1}(x)\mathbb{T}_k(x)} - \mathbb{T}_{k+1}(x)\mathbb{T}_{k-1}(x) - \cancel{2x\mathbb{T}_k(x)\mathbb{T}_{k+1}(x)} + (\mathbb{T}_k(x))^2 \\ &= (\mathbb{T}_k(x))^2 - \mathbb{T}_{k-1}(x)\mathbb{T}_{k+1}(x) \\ &= f(x) \end{aligned}$$

The first line makes use of the result (*) when $n = k$ and when $n = k + 1$.

Hence if the result is true for $n = k$ then it is true for $n = k + 1$ and as it is true for $n = 1$ it is true for all integers $n \geq 1$.

If $f(x) \equiv 0$ we have $(\mathbb{T}_n(x))^2 - \mathbb{T}_{n-1}(x)\mathbb{T}_{n+1}(x) = 0$ for all $n \geq 1$. As long as $\mathbb{T}_n(x)$ and $\mathbb{T}_{n-1}(x)$ are both non-zero we can rearrange to give:

$$\frac{\mathbb{T}_{n+1}(x)}{\mathbb{T}_n(x)} = \frac{\mathbb{T}_n(x)}{\mathbb{T}_{n-1}(x)}$$

This implies that:

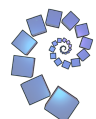
$$\frac{\mathbb{T}_n(x)}{\mathbb{T}_{n-1}(x)} = \frac{\mathbb{T}_{n-1}(x)}{\mathbb{T}_{n-2}(x)} = \dots = \frac{\mathbb{T}_1(x)}{\mathbb{T}_0(x)} = r(x)$$

And we have:

$$\begin{aligned} \mathbb{T}_n(x) &= \frac{\mathbb{T}_n(x)}{\mathbb{T}_{n-1}(x)} \times \frac{\mathbb{T}_{n-1}(x)}{\mathbb{T}_{n-2}(x)} \times \dots \times \frac{\mathbb{T}_1(x)}{\mathbb{T}_0(x)} \times \mathbb{T}_0(x) \\ &= (r(x))^n \mathbb{T}_0(x) \end{aligned}$$

Substituting this into (*) gives:

$$(r(x))^{n+1} \mathbb{T}_0(x) - 2x(r(x))^n \mathbb{T}_0(x) + (r(x))^{n-1} \mathbb{T}_0(x) = 0.$$



Since we are told to assume $T_0(x) \neq 0$ we can divide by $T_0(x)$ to get:

$$r(x)^{n-1} \left((r(x))^2 - 2x \times r(x) + 1 \right) = 0.$$

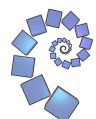
This must hold when $n = 1$, so we have:

$$(r(x))^2 - 2x \times r(x) + 1 = 0.$$

Solving the quadratic gives:

$$\begin{aligned} r(x) &= \frac{2x \pm \sqrt{4x^2 - 4}}{2} \\ &= x \pm \sqrt{x^2 - 1} \end{aligned}$$

giving two possible expressions for $r(x)$.



2 Looking at the RHS we have:

$$\begin{aligned}\sin\left(r + \frac{1}{2}\right)\theta &= \sin r\theta \cos \frac{1}{2}\theta + \cos r\theta \sin \frac{1}{2}\theta && \text{and} \\ \sin\left(r - \frac{1}{2}\right)\theta &= \sin r\theta \cos \frac{1}{2}\theta - \cos r\theta \sin \frac{1}{2}\theta\end{aligned}$$

Then subtracting the second from the first gives:

$$\sin\left(r + \frac{1}{2}\right)\theta - \sin\left(r - \frac{1}{2}\right)\theta = 2 \cos r\theta \sin \frac{1}{2}\theta$$

as required.

If

$$\cos a\theta + \cos(a+1)\theta + \cdots + \cos(b-2)\theta + \cos(b-1)\theta = 0$$

then

$$2 \sin \frac{1}{2}\theta \left(\cos a\theta + \cos(a+1)\theta + \cdots + \cos(b-2)\theta + \cos(b-1)\theta \right) = 0$$

i.e. we have:

$$\begin{aligned}\left[\sin\left(a + \frac{1}{2}\right)\theta - \sin\left(a - \frac{1}{2}\right)\theta \right] &+ \left[\sin\left(a + \frac{3}{2}\right)\theta - \sin\left(a + \frac{1}{2}\right)\theta \right] + \cdots \\ &+ \left[\sin\left(b - \frac{3}{2}\right)\theta - \sin\left(b - \frac{5}{2}\right)\theta \right] + \left[\sin\left(b - \frac{1}{2}\right)\theta - \sin\left(b - \frac{3}{2}\right)\theta \right] = 0.\end{aligned}$$

After cancelling we are left with:

$$\sin\left(b - \frac{1}{2}\right)\theta - \sin\left(a - \frac{1}{2}\right)\theta = 0.$$

It would be good if we could write this as a product, using a similar formula to the one shown at the start of the question.

Consider $\sin A - \sin B$ ¹. This can be written as

$$\begin{aligned}\sin A - \sin B &= \sin\left(\frac{1}{2}(A+B) + \frac{1}{2}(A-B)\right) - \sin\left(\frac{1}{2}(A+B) - \frac{1}{2}(A-B)\right) \\ &= \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B) + \sin \frac{1}{2}(A-B) \cos \frac{1}{2}(A+B) \\ &\quad - \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B) + \sin \frac{1}{2}(A-B) \cos \frac{1}{2}(A+B) \\ &= 2 \sin \frac{1}{2}(A-B) \cos \frac{1}{2}(A+B)\end{aligned}$$

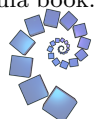
Using $A = (b - \frac{1}{2})\theta$ and $B = (a - \frac{1}{2})\theta$ gives:

$$\begin{aligned}\sin\left(b - \frac{1}{2}\right)\theta - \sin\left(a - \frac{1}{2}\right)\theta &= 0 \\ \implies 2 \times \sin \frac{1}{2}(b-a)\theta \times \cos \frac{1}{2}(b+a-1)\theta &= 0\end{aligned}$$

This means that the solutions are:

$$\begin{aligned}\frac{1}{2}(b-a)\theta &= n\pi && \text{or} \\ \frac{1}{2}(b+a-1)\theta &= \frac{(2n+1)\pi}{2}\end{aligned}$$

¹In the year that this question was set the formula for $\sin A - \sin B$ was given in the accompanying formula book.



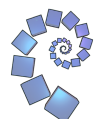
Draw a sketch to convince yourself of these general solutions for $\sin \phi = 0$ and $\cos \phi = 0$!

An alternative approach is to use $\sin A = \sin B$ if and only if $A = B + 2n\pi$ or $A = \pi - B + 2n\pi$. Draw a sketch to convince yourself why this is true!

However, we did multiply the original equation by $2 \sin \frac{1}{2}\theta$, which introduced solutions of the form $\theta = 2m\pi$. These do not solve the original equation $\cos a\theta + \cos(a+1)\theta + \dots + \cos(b-2)\theta + \cos(b-1)\theta = 0$, as all the \cos terms will be equal to 1, so we need to remove these “solutions”.

The solutions to the original equation are:

$$\begin{aligned} \frac{1}{2}(b-a)\theta = n\pi &\implies \theta = \frac{2n\pi}{b-a} \quad \text{for } n \neq k(b-a) \quad \text{and} \\ \frac{1}{2}(b+a-1)\theta = \frac{(2n+1)\pi}{2} &\implies \theta = \frac{(2n+1)\pi}{b+a-1} \end{aligned}$$



- 3** (i) Repeated use of difference of two squares will give:

$$\begin{aligned}
 (1-x)(1+x)(1+x^2)(1+x^4)\dots(1+x^{2^n}) &= (1-x^2)(1+x^2)(1+x^4)\dots(1+x^{2^n}) \\
 &= (1-x^4)\dots(1+x^{2^n}) \\
 &= (1-x^{2^n})(1+x^{2^n}) \\
 &= 1-x^{2^{n+1}}
 \end{aligned}$$

Rearranging gives:

$$\begin{aligned}
 1 &= (1-x)(1+x)(1+x^2)(1+x^4)\dots(1+x^{2^n}) + x^{2^{n+1}} \\
 \frac{1}{1-x} &= (1+x)(1+x^2)(1+x^4)\dots(1+x^{2^n}) + \frac{x^{2^{n+1}}}{1-x}.
 \end{aligned}$$

Since $|x| < 1$, as $n \rightarrow \infty$, $x^{2^{n+1}} \rightarrow 0$ and we have:

$$\begin{aligned}
 \frac{1}{1-x} &= (1+x)(1+x^2)(1+x^4)\dots(1+x^{2^r})\dots \\
 &= \prod_{r=0}^{\infty} (1+x^{2^r})
 \end{aligned}$$

Taking logs of both sides gives:

$$\begin{aligned}
 \ln\left(\frac{1}{1-x}\right) &= \ln\left(\prod_{r=0}^{\infty} (1+x^{2^r})\right) \\
 -\ln(1-x) &= \ln(1+x) + \ln(1+x^2) + \dots + \ln(1+x^{2^r}) + \dots \\
 \ln(1-x) &= -\sum_{r=0}^{\infty} \ln(1+x^{2^r}).
 \end{aligned}$$

The last part looks like it might involve differentiation. Starting with the last result we have:

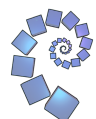
$$\begin{aligned}
 \ln(1-x) &= -\ln(1+x) - \ln(1+x^2) - \ln(1+x^4) - \dots \\
 \frac{d}{dx} \implies -\frac{1}{1-x} &= -\frac{1}{1+x} - \frac{2x}{1+x^2} - \frac{4x^3}{1+x^4} - \dots \\
 \frac{1}{1-x} &= \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \dots
 \end{aligned}$$

- (ii) Comparing this part to the previous part (note the similarities in the denominators), start by considering:

$$(1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\dots(1-x^{2^n}+x^{2^{n+1}})$$

Expanding the first two brackets gives:

$$\begin{aligned}
 (1+x+x^2)(1-x+x^2) &= (1+x^2)^2 - x^2 \\
 &= 1+2x^2+x^4 - x^2 \\
 &= 1+x^2+x^4.
 \end{aligned}$$



In general we have:

$$\begin{aligned} (1 + x^{2^r} + x^{2^{r+1}}) (1 - x^{2^r} + x^{2^{r+1}}) &= (1 + x^{2^{r+1}})^2 - (x^{2^r})^2 \\ &= 1 + 2x^{2^{r+1}} + x^{2^{r+2}} - x^{2^{r+1}} \\ &= 1 + x^{2^{r+1}} + x^{2^{r+2}} \end{aligned}$$

Using this we have:

$$(1 + x + x^2) (1 - x + x^2) (1 - x^2 + x^4) \dots (1 - x^{2^n} + x^{2^{n+1}}) = 1 + x^{2^{n+1}} + x^{2^{n+2}}.$$

and rearranging gives:

$$\frac{1}{1 + x + x^2} = (1 - x + x^2) (1 - x^2 + x^4) \dots (1 - x^{2^n} + x^{2^{n+1}}) - \left(\frac{x^{2^{n+1}} + x^{2^{n+2}}}{1 + x + x^2} \right)$$

Since $|x| < 1$ as $n \rightarrow \infty$ the last term tends to 0. We now have:

$$\frac{1}{1 + x + x^2} = \prod_{r=0}^{\infty} (1 - x^{2^r} + x^{2^{r+1}})$$

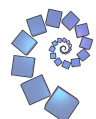
Taking logs of both sides results in:

$$\begin{aligned} -\ln(1 + x + x^2) &= \sum_{r=0}^{\infty} \ln(1 - x^{2^r} + x^{2^{r+1}}) \\ \ln(1 + x + x^2) &= -\sum_{r=0}^{\infty} \ln(1 - x^{2^r} + x^{2^{r+1}}) \\ \ln(1 + x + x^2) &= -\ln(1 - x + x^2) - \ln(1 - x^2 + x^4) - \ln(1 - x^4 + x^8) \\ &\quad - \ln(1 - x^8 + x^{16}) - \dots \end{aligned}$$

Then differentiating both sides gives:

$$\begin{aligned} \frac{1 + 2x}{(1 + x + x^2)} &= -\frac{-1 + 2x}{(1 - x + x^2)} - \frac{-2x + 4x^3}{(1 - x^2 + x^4)} - \frac{-4x^3 + 8x^7}{(1 - x^4 + x^8)} - \dots \\ &= \frac{1 - 2x}{(1 - x + x^2)} + \frac{2x - 4x^3}{(1 - x^2 + x^4)} + \frac{4x^3 - 8x^7}{(1 - x^4 + x^8)} + \dots \end{aligned}$$

Alternatively, you could replace x by x^3 in the result $\ln(1 - x) = -\sum_{r=0}^{\infty} \ln(1 + x^{2^r})$ from part (i), then use the difference and sum of two cubes formulae, $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ and $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, and subtract the part (i) result before differentiating. Both ways are fine!



4 (i) If α is a root of both equations then we have:

$$\alpha^2 + a\alpha + b = 0 \quad \text{and} \quad (1)$$

$$\alpha^2 + c\alpha + d = 0 \quad (2)$$

Evaluating (1) – (2) gives:

$$\alpha(a - c) + (b - d) = 0$$

$$\alpha(a - c) = -(b - d)$$

$$\alpha = -\frac{b - d}{a - c} \quad \text{as } a - c \neq 0$$

Starting with the “if” part we have:

$$(b - d)^2 - a(b - d)(a - c) + b(a - c)^2 = 0 \quad \text{divide by } (a - c)^2$$

$$\left(\frac{b - d}{a - c}\right)^2 - a\left(\frac{b - d}{a - c}\right) + b = 0$$

and so $x = -\frac{b-d}{a-c}$ is a solution of $x^2 + ax + b = 0$. Substituting into the other equation gives:

$$\begin{aligned} x^2 + cx + d &= \left(\frac{b - d}{a - c}\right)^2 + c\left(-\frac{b - d}{a - c}\right) + d \\ &= \left(\frac{b - d}{a - c}\right)^2 + (c - a)\left(-\frac{b - d}{a - c}\right) + a\left(-\frac{b - d}{a - c}\right) + (d - b) + b \\ &= \left[\left(\frac{b - d}{a - c}\right)^2 + a\left(-\frac{b - d}{a - c}\right) + b\right] + (c - a)\left(-\frac{b - d}{a - c}\right) + (d - b) \\ &= [0] + \cancel{(a - c)}\frac{b - d}{\cancel{a - c}} + (d - b) \\ &= 0 \end{aligned}$$

So $x = -\frac{b-d}{a-c}$ is a solution of both equations.

Going the other way (“Only if”), if the equations have a common root, α then we have $\alpha = -\frac{b-d}{a-c}$. Substituting into $x^2 + ax + b = 0$ gives:

$$\left(-\frac{b - d}{a - c}\right)^2 + a\left(-\frac{b - d}{a - c}\right) + b = 0 \quad \text{multiply by } (a - c)^2$$

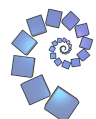
$$(b - d)^2 - a(b - d)(a - c) + b(a - c)^2 = 0$$

Hence the equations have at least one common root if and only if

$$(b - d)^2 - a(b - d)(a - c) + b(a - c)^2 = 0.$$

If we have $(b - d)^2 - a(b - d)(a - c) + b(a - c)^2 = 0$ and $a = c$ then this implies that $b - d = 0$ and hence $b = d$ and the two equations are the same (therefore must have at least one common root!).

If we have at least one common root, α , and $a = c$, then we have $\alpha^2 + a\alpha + b = 0$ and $\alpha^2 + a\alpha + d = 0$ which implies that $b = d$ and hence $(b - d)^2 - a(b - d)(a - c) + b(a - c)^2 = 0$ and so the result still holds when $a = c$.



(ii) If

$$(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0$$

then letting $d \rightarrow r$ and $c \rightarrow q-b$ from part (i) shows that

$$x^2 + ax + b = 0 \quad \text{and} \quad (3)$$

$$x^2 + (q-b)x + r = 0 \quad \text{have a common root.} \quad (4)$$

If α is a common root to (3) and (4), then it is also a root of $x \times (3) + (4)$, i.e.:

$$x(x^2 + ax + b) + x^2 + (q-b)x + r = 0$$

$$x^3 + ax^2 + bx + x^2 + qx - bx + r = 0$$

$$x^3 + (a+1)x^2 + qx + r = 0$$

Therefore, if $(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0$ then $x^2 + ax + b = 0$ and $x^3 + (a+1)x^2 + cx + d = 0$ have at least one common root.

If $x^2 + ax + b = 0$ and $x^3 + (a+1)x^2 + qx + r = 0$ have a common root α then we have:

$$\alpha^2 + a\alpha + b = 0 \quad (5)$$

$$\alpha^3 + (a+1)\alpha^2 + q\alpha + r = 0 \quad (6)$$

Then considering (6) - $\alpha(5)$ we have:

$$\alpha^3 + (a+1)\alpha^2 + q\alpha + r - \alpha(\alpha^2 + a\alpha + b) = 0$$

$$\cancel{\alpha^3} + (a+1)\alpha^2 + q\alpha + r - \cancel{\alpha^3} - a\alpha^2 - b\alpha = 0$$

$$\alpha^2 + (q-b)\alpha + r = 0$$

Hence the two equations $x^2 + ax + b = 0$ and $x^2 + (q-b)x + r = 0$ have at least one common root, and so by using the result in part (i) we have:

$$\begin{aligned} (b-d)^2 - a(b-d)(a-c) + b(a-c)^2 &= 0 & \text{let } c = q-b \text{ and } d = r \\ (b-r)^2 - a(b-r)(a-q+b) + b(a-q+b)^2 &= 0 & \text{as required.} \end{aligned}$$

For the last part, take $a = \frac{5}{2}$, $q = \frac{5}{2}$ and $r = \frac{1}{2}$, and then we have:

$$(b-r)^2 - a(b-r)(a-q+b) + b(a-q+b)^2 = 0$$

$$\left(b - \frac{1}{2}\right)^2 - \frac{5}{2}\left(b - \frac{1}{2}\right)\left(\frac{5}{2} - \frac{5}{2} + b\right) + b\left(\frac{5}{2} - \frac{5}{2} + b\right)^2 = 0$$

$$b^2 - b + \frac{1}{4} - \frac{5}{2}b\left(b - \frac{1}{2}\right) + b^3 = 0$$

$$b^3 + b^2 - b + \frac{1}{4} - \frac{5}{2}b^2 + \frac{5}{4}b = 0$$

$$b^3 - \frac{3}{2}b^2 + \frac{1}{4}b + \frac{1}{4} = 0$$

$$4b^3 - 6b^2 + b + 1 = 0$$

One of the solutions is $b = 1$, so the others are given by $4b^2 - 2b - 1 = 0$, i.e.

$$b = \frac{2 \pm \sqrt{20}}{8} = \frac{1 \pm \sqrt{5}}{4}.$$

