STEP Support Programme

STEP 3 Algebra: Solutions

1 **Base Case** Letting $n = 1$ we have:

$$\begin{aligned}(\mathrm{T}_n(x))^2 - \mathrm{T}_{n-1}(x)\mathrm{T}_{n+1}(x) &= \mathrm{f}(x)\\ (\mathrm{T}_1(x))^2 - \mathrm{T}_0(x)\mathrm{T}_2(x) &= \mathrm{f}(x)\end{aligned}$$

Which is true for the given $\mathrm{f}(x)$, hence the result is true when $n = 1$.

Inductive Step Assume the result is true when $n = k$, i.e. we have:

$$(\mathrm{T}_k(x))^2 - \mathrm{T}_{k-1}(x)\mathrm{T}_{k+1}(x) = \mathrm{f}(x)$$

Now consider the LHS when $n = k + 1$.

$$\begin{aligned}(\mathrm{T}_{k+1}(x))^2 - \mathrm{T}_k(x)\mathrm{T}_{k+2}(x) &= \mathrm{T}_{k+1}(x)\Big(2x\mathrm{T}_k(x) - \mathrm{T}_{k-1}(x)\Big) - \mathrm{T}_k(x)\Big(2x\mathrm{T}_{k+1}(x) - \mathrm{T}_k(x)\Big)\\ &= \cancel{2x\mathrm{T}_{k+1}(x)\mathrm{T}_k(x)} - \mathrm{T}_{k+1}(x)\mathrm{T}_{k-1}(x) - \cancel{2x\mathrm{T}_k(x)\mathrm{T}_{k+1}(x)} + (\mathrm{T}_k(x))^2\\ &= (\mathrm{T}_k(x))^2 - \mathrm{T}_{k-1}(x)\mathrm{T}_{k+1}(x)\\ &= \mathrm{f}(x)\end{aligned}$$

The first line makes use of the result $(*)$ when $n = k$ and when $n = k + 1$.

Hence if the result is true for $n = k$ then it is true for $n = k + 1$ and as it is true for $n = 1$ it is true for all integers $n \geqslant 1$.

If $\mathrm{f}(x) \equiv 0$ we have $(\mathrm{T}_n(x))^2 - \mathrm{T}_{n-1}(x)\mathrm{T}_{n+1}(x) = 0$ for all $n \geqslant 1$. As long as $\mathrm{T}_n(x)$ and $\mathrm{T}_{n-1}(x)$ are both non-zero we can rearrange to give:

$$\frac{\mathrm{T}_{n+1}(x)}{\mathrm{T}_n(x)} = \frac{\mathrm{T}_n(x)}{\mathrm{T}_{n-1}(x)}$$

This implies that:

$$\frac{\mathrm{T}_n(x)}{\mathrm{T}_{n-1}(x)} = \frac{\mathrm{T}_{n-1}(x)}{\mathrm{T}_{n-2}(x)} = \ldots = \frac{\mathrm{T}_1(x)}{\mathrm{T}_0(x)} = \mathrm{r}(x)$$

And we have:

$$\begin{aligned}\mathrm{T}_n(x) &= \frac{\mathrm{T}_n(x)}{\mathrm{T}_{n-1}(x)} \times \frac{\mathrm{T}_{n-1}(x)}{\mathrm{T}_{n-2}(x)} \times \ldots \times \frac{\mathrm{T}_1(x)}{\mathrm{T}_0(x)} \times \mathrm{T}_0(x)\\ &= \big(\mathrm{r}(x)\big)^n\,\mathrm{T}_0(x)\end{aligned}$$

Substituting this into $(*)$ gives:

$$\big(\mathrm{r}(x)\big)^{n+1}\,\mathrm{T}_0(x) - 2x\big(\mathrm{r}(x)\big)^n\,\mathrm{T}_0(x) + \big(\mathrm{r}(x)\big)^{n-1}\,\mathrm{T}_0(x) = 0\,.$$

Since we are told to assume $\mathrm{T}_0(x) \neq 0$ we can divide by $\mathrm{T}_0(x)$ to get:

$$\mathrm{r}(x)^{n-1}\Big(\left(\mathrm{r}(x)\right)^2 - 2x \times \mathrm{r}(x) + 1\Big) = 0\,.$$

This must hold when $n = 1$, so we have:

$$\left(\mathrm{r}(x)\right)^2 - 2x \times \mathrm{r}(x) + 1 = 0\,.$$

Solving the quadratic gives:

$$\begin{aligned}\mathrm{r}(x) &= \frac{2x \pm \sqrt{4x^2 - 4}}{2} \\ &= x \pm \sqrt{x^2 - 1}\end{aligned}$$

giving two possible expressions for $\mathrm{r}(x)$.

2 Looking at the RHS we have:

$$\sin\left(r+\tfrac{1}{2}\right)\theta = \sin r\theta\cos\tfrac{1}{2}\theta + \cos r\theta\sin\tfrac{1}{2}\theta \qquad \text{and}$$
$$\sin\left(r-\tfrac{1}{2}\right)\theta = \sin r\theta\cos\tfrac{1}{2}\theta - \cos r\theta\sin\tfrac{1}{2}\theta$$

Then subtracting the second from the first gives:

$$\sin\left(r+\tfrac{1}{2}\right)\theta - \sin\left(r-\tfrac{1}{2}\right)\theta = 2\cos r\theta\sin\tfrac{1}{2}\theta$$

as required.

If

$$\cos a\theta + \cos(a+1)\theta + \cdots + \cos(b-2)\theta + \cos(b-1)\theta = 0$$

then

$$2\sin\tfrac{1}{2}\theta\Big(\cos a\theta + \cos(a+1)\theta + \cdots + \cos(b-2)\theta + \cos(b-1)\theta\Big) = 0$$

i.e. we have:

$$\left[\sin\left(a+\tfrac{1}{2}\right)\theta - \sin\left(a-\tfrac{1}{2}\right)\theta\right] + \left[\sin\left(a+\tfrac{3}{2}\right)\theta - \sin\left(a+\tfrac{1}{2}\right)\theta\right] + \ldots$$
$$+\left[\sin\left(b-\tfrac{3}{2}\right)\theta - \sin\left(b-\tfrac{5}{2}\right)\theta\right] + \left[\sin\left(b-\tfrac{1}{2}\right)\theta - \sin\left(b-\tfrac{3}{2}\right)\theta\right] = 0\,.$$

After cancelling we are left with:

$$\sin\left(b-\tfrac{1}{2}\right)\theta - \sin\left(a-\tfrac{1}{2}\right)\theta = 0\,.$$

It would be good if we could write this as a product, using a similar formula to the one shown at the start of the question.

Consider $\sin A - \sin B$[1]. This can be written as

$$\begin{aligned}\sin A - \sin B = \ & \sin\Big(\tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B)\Big) - \sin\Big(\tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B)\Big)\\ = \ & \sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) + \sin\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B)\\ & -\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) + \sin\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B)\\ =& 2\sin\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B)\end{aligned}$$

Using $A = (b-\frac{1}{2})\theta$ and $B = (a-\frac{1}{2})\theta$ gives:

$$\begin{aligned}\sin\left(b-\tfrac{1}{2}\right)\theta - \sin\left(a-\tfrac{1}{2}\right)\theta &= 0\\ \implies 2\times\sin\tfrac{1}{2}(b-a)\theta\times\cos\tfrac{1}{2}(b+a-1)\theta &= 0\end{aligned}$$

This means that the solutions are:

$$\tfrac{1}{2}(b-a)\theta = n\pi \qquad \text{or}$$
$$\tfrac{1}{2}(b+a-1)\theta = \frac{(2n+1)\pi}{2}$$

[1] In the year that this question was set the formula for $\sin A - \sin B$ was given in the accompanying formula book.

Draw a sketch to convince yourself of these general solutions for $\sin\phi = 0$ and $\cos\phi = 0$!

An alternative approach is to use $\sin A = \sin B$ if and only if $A = B + 2n\pi$ or $A = \pi - B + 2n\pi$. Draw a sketch to convince yourself why this is true!

However, we did multiply the original equation by $2\sin\frac{1}{2}\theta$, which introduced solutions of the form $\theta = 2m\pi$. These do not solve the original equation $\cos a\theta + \cos(a+1)\theta + \cdots + \cos(b-2)\theta + \cos(b-1)\theta = 0$, as all the cos terms will be equal to 1, so we need to remove these "solutions".

The solutions to the original equation are:

$$\tfrac{1}{2}(b-a)\theta = n\pi \implies \theta = \frac{2n\pi}{b-a} \qquad \text{for } n \neq k(b-a) \qquad \text{and}$$
$$\tfrac{1}{2}(b+a-1)\theta = \frac{(2n+1)\pi}{2} \implies \theta = \frac{(2n+1)\pi}{b+a-1}$$

3 **(i)** Repeated use of difference of two squares will give:

$$\begin{aligned}(1-x)(1+x)\left(1+x^2\right)\left(1+x^4\right)\dots\left(1+x^{2^n}\right) &= \left(1-x^2\right)\left(1+x^2\right)\left(1+x^4\right)\dots\left(1+x^{2^n}\right)\\ &= \left(1-x^4\right)\dots\left(1+x^{2^n}\right)\\ &= \left(1-x^{2^n}\right)\left(1+x^{2^n}\right)\\ &= 1-x^{2^{n+1}}\end{aligned}$$

Rearranging gives:

$$\begin{aligned}1 &= (1-x)(1+x)\left(1+x^2\right)\left(1+x^4\right)\dots\left(1+x^{2^n}\right)+x^{2^{n+1}}\\ \frac{1}{1-x} &= (1+x)\left(1+x^2\right)\left(1+x^4\right)\dots\left(1+x^{2^n}\right)+\frac{x^{2^{n+1}}}{1-x}\,.\end{aligned}$$

Since $|x|<1$, as $n\to\infty$, $x^{2^{n+1}}\to 0$ and we have:

$$\begin{aligned}\frac{1}{1-x} &= (1+x)\left(1+x^2\right)\left(1+x^4\right)\dots\left(1+x^{2^r}\right)\dots\\ &= \prod_{r=0}^{\infty}\left(1+x^{2^r}\right)\end{aligned}$$

Taking logs of both sides gives:

$$\begin{aligned}\ln\left(\frac{1}{1-x}\right) &= \ln\left(\prod_{r=0}^{\infty}\left(1+x^{2^r}\right)\right)\\ -\ln(1-x) &= \ln(1+x)+\ln\left(1+x^2\right)+\dots+\ln\left(1+x^{2^r}\right)+\dots\\ \ln(1-x) &= -\sum_{r=0}^{\infty}\ln\left(1+x^{2^r}\right)\,.\end{aligned}$$

The last part looks like it might involve differentiation. Starting with the last result we have:

$$\begin{aligned} & \ln(1-x) = -\ln(1+x)-\ln\left(1+x^2\right)-\ln\left(1+x^4\right)-\dots\\ \frac{\mathrm{d}}{\mathrm{d}x}\implies\quad & -\frac{1}{1-x} = -\frac{1}{1+x}-\frac{2x}{1+x^2}-\frac{4x^3}{1+x^4}-\dots\\ & \frac{1}{1-x} = \frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\dots\end{aligned}$$

(ii) Comparing this part to the previous part (note the similarities in the denominators), start by considering:

$$\left(1+x+x^2\right)\left(1-x+x^2\right)\left(1-x^2+x^4\right)\left(1-x^4+x^8\right)\dots\left(1-x^{2^n}+x^{2^{n+1}}\right)$$

Expanding the first two brackets gives:

$$\begin{aligned}\left(1+x+x^2\right)\left(1-x+x^2\right) &= \left(1+x^2\right)^2-x^2\\ &= 1+2x^2+x^4-x^2\\ &= 1+x^2+x^4\,.\end{aligned}$$

In general we have:

$$\begin{aligned}\left(1+x^{2^r}+x^{2^{r+1}}\right)\left(1-x^{2^r}+x^{2^{r+1}}\right) &= \left(1+x^{2^{r+1}}\right)^2-\left(x^{2^r}\right)^2\\ &= 1+2x^{2^{r+1}}+x^{2^{r+2}}-x^{2^{r+1}}\\ &= 1+x^{2^{r+1}}+x^{2^{r+2}}\end{aligned}$$

Using this we have:

$$\left(1+x+x^2\right)\left(1-x+x^2\right)\left(1-x^2+x^4\right)\dots\left(1-x^{2^n}+x^{2^{n+1}}\right)=1+x^{2^{n+1}}+x^{2^{n+2}}\,.$$

and rearranging gives:

$$\frac{1}{1+x+x^2}=\left(1-x+x^2\right)\left(1-x^2+x^4\right)\dots\left(1-x^{2^n}+x^{2^{n+1}}\right)-\left(\frac{x^{2^{n+1}}+x^{2^{n+2}}}{1+x+x^2}\right)$$

Since $|x|<1$ as $n\to\infty$ the last term tends to 0. We now have:

$$\frac{1}{1+x+x^2}=\prod_{r=0}^{\infty}\left(1-x^{2^r}+x^{2^{r+1}}\right)$$

Taking logs of both sides results in:

$$\begin{aligned}-\ln\left(1+x+x^2\right) &= \sum_{r=0}^{\infty}\ln\left(1-x^{2^r}+x^{2^{r+1}}\right)\\ \ln\left(1+x+x^2\right) &= -\sum_{r=0}^{\infty}\ln\left(1-x^{2^r}+x^{2^{r+1}}\right)\\ \ln\left(1+x+x^2\right) &= -\ln\left(1-x+x^2\right)-\ln\left(1-x^2+x^4\right)-\ln\left(1-x^4+x^8\right)\\ &\qquad -\ln\left(1-x^8+x^{16}\right)-\dots\end{aligned}$$

Then differentiating both sides gives:

$$\begin{aligned}\frac{1+2x}{(1+x+x^2)} &= -\frac{-1+2x}{(1-x+x^2)}-\frac{-2x+4x^3}{(1-x^2+x^4)}-\frac{-4x^3+8x^7}{(1-x^4+x^8)}-\dots\\ &= \frac{1-2x}{(1-x+x^2)}+\frac{2x-4x^3}{(1-x^2+x^4)}+\frac{4x^3-8x^7}{(1-x^4+x^8)}+\dots\end{aligned}$$

Alternatively, you could replace x by x^3 in the result $\ln(1-x)=-\sum_{r=0}^{\infty}\ln\left(1+x^{2^r}\right)$ from part **(i)**, then use the difference and sum of two cubes formulae, $a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$ and $a^3+b^3=(a+b)\left(a^2-ab+b^2\right)$, and subtract the part **(i)** result before differentiating. Both ways are fine!

4 **(i)** If α is a root of both equations then we have:

$$\alpha^2 + a\alpha + b = 0 \qquad \text{and} \qquad (1)$$
$$\alpha^2 + c\alpha + d = 0 \qquad (2)$$

Evaluating (1) − (2) gives:

$$\begin{aligned} \alpha(a-c) + (b-d) &= 0 \\ \alpha(a-c) &= -(b-d) \\ \alpha &= -\frac{b-d}{a-c} \qquad \text{as } a-c \neq 0 \end{aligned}$$

Starting with the "if" part we have:

$$\begin{aligned} (b-d)^2 - a(b-d)(a-c) + b(a-c)^2 &= 0 \qquad \text{divide by } (a-c)^2 \\ \left(\frac{b-d}{a-c}\right)^2 - a\left(\frac{b-d}{a-c}\right) + b &= 0 \end{aligned}$$

and so $x = -\frac{b-d}{a-c}$ is a solution of $x^2 + ax + b = 0$. Substituting into the other equation gives:

$$\begin{aligned} x^2 + cx + d &= \left(\frac{b-d}{a-c}\right)^2 + c\left(-\frac{b-d}{a-c}\right) + d \\ &= \left(\frac{b-d}{a-c}\right)^2 + (c-a)\left(-\frac{b-d}{a-c}\right) + a\left(-\frac{b-d}{a-c}\right) + (d-b) + b \\ &= \left[\left(\frac{b-d}{a-c}\right)^2 + a\left(-\frac{b-d}{a-c}\right) + b\right] + (c-a)\left(-\frac{b-d}{a-c}\right) + (d-b) \\ &= \left[0\right] + \cancel{(a-c)}\frac{b-d}{\cancel{a-c}} + (d-b) \\ &= 0 \end{aligned}$$

So $x = -\frac{b-d}{a-c}$ is a solution of both equations.

Going the other way ("Only if"), if the equations have a common root, α then we have $\alpha = -\frac{b-d}{a-c}$. Substituting into $x^2 + ax + b = 0$ gives:

$$\begin{aligned} \left(-\frac{b-d}{a-c}\right)^2 + a\left(-\frac{b-d}{a-c}\right) + b &= 0 \qquad \text{multiply by } (a-c)^2 \\ (b-d)^2 - a(b-d)(a-c) + b(a-c)^2 &= 0 \end{aligned}$$

Hence the equations have at least one common root if and only if
$(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0$.

If we have $(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0$ and $a = c$ then this implies that $b - d = 0$ and hence $b = d$ and the two equations are the same (therefore must have at least one common root!).

If we have at least one common root, α, and $a = c$, then we have $\alpha^2 + a\alpha + b = 0$ and $\alpha^2 + a\alpha + d = 0$ which implies that $b = d$ and hence $(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0$ and so the result still holds when $a = c$.

(ii) If

$$(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0$$

then letting $d \to r$ and $c \to q-b$ from part **(i)** shows that

$$x^2 + ax + b = 0 \qquad \text{and} \tag{3}$$
$$x^2 + (q-b)x + r = 0 \qquad \text{have a common root.} \tag{4}$$

If α is a common root to (3) and (4), then it is also a root of $x \times (3) + (4)$, i.e.:

$$\begin{aligned} x\left(x^2+ax+b\right) + x^2 + (q-b)x + r &= 0 \\ x^3 + ax^2 + bx + x^2 + qx - bx + r &= 0 \\ x^3 + (a+1)x^2 + qx + r &= 0 \end{aligned}$$

Therefore, if $(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0$ then $x^2+ax+b=0$ and $x^3+(a+1)x^2+cx+d=0$ have at least one common root.

If $x^2+ax+b=0$ and $x^3+(a+1)x^2+qx+r=0$ have a common root α then we have:

$$\alpha^2 + a\alpha + b = 0 \tag{5}$$
$$\alpha^3 + (a+1)\alpha^2 + q\alpha + r = 0 \tag{6}$$

Then considering $(6) - \alpha(5)$ we have:

$$\begin{aligned} \alpha^3 + (a+1)\alpha^2 + q\alpha + r - \alpha\left(\alpha^2 + a\alpha + b\right) &= 0 \\ \cancel{\alpha^3} + (a+1)\alpha^2 + q\alpha + r - \cancel{\alpha^3} - a\alpha^2 - b\alpha &= 0 \\ \alpha^2 + (q-b)\alpha + r &= 0 \end{aligned}$$

Hence the two equations $x^2+ax+b=0$ and $x^2+(q-b)x+r=0$ have at least one common root, and so by using the result in part **(i)** we have:

$$\begin{aligned} (b-d)^2 - a(b-d)(a-c) + b(a-c)^2 &= 0 \qquad \text{let } c = q-b \text{ and } d = r \\ (b-r)^2 - a(b-r)(a-q+b) + b(a-q+b)^2 &= 0 \qquad \text{as required.} \end{aligned}$$

For the last part, take $a=\frac{5}{2}$, $q=\frac{5}{2}$ and $r=\frac{1}{2}$, and then we have:

$$\begin{aligned} (b-r)^2 - a(b-r)(a-q+b) + b(a-q+b)^2 &= 0 \\ \left(b-\tfrac{1}{2}\right)^2 - \tfrac{5}{2}\left(b-\tfrac{1}{2}\right)\left(\tfrac{5}{2}-\tfrac{5}{2}+b\right) + b\left(\tfrac{5}{2}-\tfrac{5}{2}+b\right)^2 &= 0 \\ b^2 - b + \tfrac{1}{4} - \tfrac{5}{2}b\left(b-\tfrac{1}{2}\right) + b^3 &= 0 \\ b^3 + b^2 - b + \tfrac{1}{4} - \tfrac{5}{2}b^2 + \tfrac{5}{4}b &= 0 \\ b^3 - \tfrac{3}{2}b^2 + \tfrac{1}{4}b + \tfrac{1}{4} &= 0 \\ 4b^3 - 6b^2 + b + 1 &= 0 \end{aligned}$$

One of the solutions is $b=1$, so the others are given by $4b^2-2b-1=0$, i.e. $b = \dfrac{2 \pm \sqrt{20}}{8} = \dfrac{1 \pm \sqrt{5}}{4}$.