STEP Support Programme

STEP 3 Algebra: Solutions

1 Base Case Letting \( n = 1 \) we have:

\[
(T_n(x))^2 - T_{n-1}(x)T_{n+1}(x) = f(x)
\]
\[
(T_1(x))^2 - T_0(x)T_2(x) = f(x)
\]

Which is true for the given \( f(x) \), hence the result is true when \( n = 1 \).

Inductive Step Assume the result is true when \( n = k \), i.e. we have:

\[
(T_k(x))^2 - T_{k-1}(x)T_{k+1}(x) = f(x)
\]

Now consider the LHS when \( n = k + 1 \).

\[
(T_{k+1}(x))^2 - T_k(x)T_{k+2}(x) = T_{k+1}(x)\left(2xT_k(x) - T_{k-1}(x)\right) - T_k(x)\left(2xT_{k+1}(x) - T_k(x)\right)
\]
\[
= 2xT_{k+1}(x)T_k(x) - T_{k+1}(x)T_{k-1}(x) - 2xT_k(x)T_{k+1}(x) + (T_k(x))^2
\]
\[
= (T_k(x))^2 - T_{k-1}(x)T_{k+1}(x)
\]
\[
= f(x)
\]

The first line makes use of the result (*) when \( n = k \) and when \( n = k + 1 \).

Hence if the result is true for \( n = k \) then it is true for \( n = k + 1 \) and as it is true for \( n = 1 \) it is true for all integers \( n \geq 1 \).

If \( f(x) \equiv 0 \) we have \( (T_n(x))^2 - T_{n-1}(x)T_{n+1}(x) = 0 \) for all \( n \geq 1 \). As long as \( T_n(x) \) and \( T_{n-1}(x) \) are both non-zero we can rearrange to give:

\[
\frac{T_{n+1}(x)}{T_n(x)} = \frac{T_n(x)}{T_{n-1}(x)}
\]

This implies that:

\[
\frac{T_n(x)}{T_{n-1}(x)} = \frac{T_{n-1}(x)}{T_{n-2}(x)} = \ldots = \frac{T_1(x)}{T_0(x)} = r(x)
\]

And we have:

\[
T_n(x) = \frac{T_n(x)}{T_{n-1}(x)} \times \frac{T_{n-1}(x)}{T_{n-2}(x)} \times \ldots \times \frac{T_1(x)}{T_0(x)} \times T_0(x)
\]
\[
= (r(x))^n T_0(x)
\]

Substituting this into (*) gives:

\[
(r(x))^{n+1} T_0(x) - 2x(r(x))^n T_0(x) + (r(x))^{n-1} T_0(x) = 0.
\]
Since we are told to assume $T_0(x) \neq 0$ we can divide by $T_0(x)$ to get:

$$r(x)^{n-1} \left( (r(x))^2 - 2x \times r(x) + 1 \right) = 0.$$  

This must hold when $n = 1$, so we have:

$$(r(x))^2 - 2x \times r(x) + 1 = 0.$$  

Solving the quadratic gives:

$$r(x) = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$= x \pm \sqrt{x^2 - 1}$$

giving two possible expressions for $r(x)$.  

Looking at the RHS we have:

\[
\sin \left( r + \frac{1}{2} \right) \theta = \sin r \theta \cos \frac{1}{2} \theta + \cos r \theta \sin \frac{1}{2} \theta \quad \text{and} \quad \sin \left( r - \frac{1}{2} \right) \theta = \sin r \theta \cos \frac{1}{2} \theta - \cos r \theta \sin \frac{1}{2} \theta
\]

Then subtracting the second from the first gives:

\[
\sin \left( r + \frac{1}{2} \right) \theta - \sin \left( r - \frac{1}{2} \right) \theta = 2 \cos r \theta \sin \frac{1}{2} \theta
\]

as required.

If

\[
\cos a \theta + \cos(a + 1) \theta + \cdots + \cos(b - 2) \theta + \cos(b - 1) \theta = 0
\]

then

\[
2 \sin \frac{1}{2} \theta \left( \cos a \theta + \cos(a + 1) \theta + \cdots + \cos(b - 2) \theta + \cos(b - 1) \theta \right) = 0
\]

i.e. we have:

\[
\left[ \sin \left( a + \frac{1}{2} \right) \theta - \sin \left( a - \frac{1}{2} \right) \theta \right] + \left[ \sin \left( a + \frac{3}{2} \right) \theta - \sin \left( a + \frac{1}{2} \right) \theta \right] + \cdots
\]

\[
+ \left[ \sin \left( b - \frac{3}{2} \right) \theta - \sin \left( b - \frac{5}{2} \right) \theta \right] + \left[ \sin \left( b - \frac{1}{2} \right) \theta - \sin \left( b - \frac{3}{2} \right) \theta \right] = 0.
\]

After cancelling we are left with:

\[
\sin \left( b - \frac{1}{2} \right) \theta - \sin \left( a - \frac{1}{2} \right) \theta = 0.
\]

It would be good if we could write this as a product, using a similar formula to the one shown at the start of the question.

Consider \( \sin A - \sin B \). This can be written as

\[
\sin A - \sin B = \sin \left( \frac{1}{2}(A + B) + \frac{1}{2}(A - B) \right) - \sin \left( \frac{1}{2}(A + B) - \frac{1}{2}(A - B) \right)
\]

\[
= \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B) + \sin \frac{1}{2}(A - B) \cos \frac{1}{2}(A + B)
\]

\[
- \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B) - \sin \frac{1}{2}(A - B) \cos \frac{1}{2}(A + B)
\]

\[
= 2 \sin \frac{1}{2}(A - B) \cos \frac{1}{2}(A + B)
\]

Using \( A = (b - \frac{1}{2}) \theta \) and \( B = (a - \frac{1}{2}) \theta \) gives:

\[
\sin \left( b - \frac{1}{2} \right) \theta - \sin \left( a - \frac{1}{2} \right) \theta = 0
\]

\[
\implies 2 \times \sin \frac{1}{2}(b - a) \theta \times \cos \frac{1}{2}(b + a - 1) \theta = 0
\]

This means that the solutions are:

\[
\frac{1}{2}(b - a) \theta = n\pi \quad \text{or} \quad \frac{1}{2}(b + a - 1) \theta = \frac{(2n + 1)\pi}{2}
\]

\[1\text{In the year that this question was set the formula for } \sin A - \sin B \text{ was given in the accompanying formula book.}\]
Draw a sketch to convince yourself of these general solutions for \( \sin \phi = 0 \) and \( \cos \phi = 0 \! \)!

An alternative approach is to use \( \sin A = \sin B \) if and only if \( A = B + 2n\pi \) or \( A = \pi - B + 2n\pi \). Draw a sketch to convince yourself why this is true!

However, we did multiply the original equation by \( 2 \sin \frac{1}{2}\theta \), which introduced solutions of the form \( \theta = 2m\pi \). These do not solve the original equation \( \cos a\theta + \cos(a+1)\theta + \cdots + \cos(b-2)\theta + \cos(b-1)\theta = 0 \), as all the \( \cos \) terms will be equal to 1, so we need to remove these “solutions”.

The solutions to the original equation are:

\[
\frac{1}{2}(b-a)\theta = n\pi \quad \Rightarrow \quad \theta = \frac{2n\pi}{b-a} \quad \text{for} \ n \neq k(b-a) \quad \text{and} \\
\frac{1}{2}(b+a-1)\theta = \frac{(2n+1)\pi}{2} \quad \Rightarrow \quad \theta = \frac{(2n+1)\pi}{b+a-1}
\]
3 (i) Repeated use of difference of two squares will give:

\[(1 - x)(1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{2n}) = (1 - x^2)(1 + x^2)(1 + x^4) \ldots (1 + x^{2n})
= (1 - x^4) \ldots (1 + x^{2n})
= (1 - x^{2n})(1 + x^{2n})
= 1 - x^{2n+1}\]

Rearranging gives:

\[1 = (1 - x)(1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{2n}) + x^{2n+1}
\]

\[\frac{1}{1 - x} = (1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{2n}) + \frac{x^{2n+1}}{1 - x}.\]

Since \(|x| < 1\), as \(n \to \infty\), \(x^{2n+1} \to 0\) and we have:

\[\frac{1}{1 - x} = (1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{2n}) \ldots
= \prod_{r=0}^{\infty} (1 + x^{2r})\]

Taking logs of both sides gives:

\[\ln \left( \frac{1}{1 - x} \right) = \ln \left( \prod_{r=0}^{\infty} (1 + x^{2r}) \right)
- \ln(1 - x) = \ln(1 + x) + \ln \left( 1 + x^2 \right) + \ldots + \ln \left( 1 + x^{2r} \right) + \ldots
\]

\[\ln(1 - x) = - \sum_{r=0}^{\infty} \ln \left( 1 + x^{2r} \right).\]

The last part looks like it might involve differentiation. Starting with the last result
we have:

\[\frac{d}{dx} \ln(1 - x) = - \ln(1 + x) - \ln \left( 1 + x^2 \right) - \ln \left( 1 + x^4 \right) - \ldots\]

\[\frac{d}{dx} \Rightarrow - \frac{1}{1 - x} = - \frac{1}{1 + x} - \frac{2x}{1 + x^2} - \frac{4x^3}{1 + x^4} - \ldots
\]

\[\frac{1}{1 - x} = \frac{1}{1 + x} + \frac{2x}{1 + x^2} + \frac{4x^3}{1 + x^4} + \ldots\]

(ii) Comparing this part to the previous part (note the similarities in the denominators),
start by considering:

\[(1 + x + x^2)(1 - x + x^2)(1 - x^2 + x^4)(1 - x^4 + x^8) \ldots \left(1 - x^{2n} + x^{2n+1}\right)\]

Expanding the first two brackets gives:

\[(1 + x + x^2)(1 - x + x^2) = (1 + x^2)^2 - x^2
= 1 + 2x^2 + x^4 - x^2
= 1 + x^2 + x^4.\]
In general we have:

\[
\left(1 + x^{2r} + x^{2r+1}\right)\left(1 - x^{2r} + x^{2r+1}\right) = \left(1 + x^{2r+1}\right)^2 - (x^{2r})^2 \\
= 1 + 2x^{2r+1} + x^{2r+2} - x^{4r+1} \\
= 1 + x^{2r+1} + x^{2r+2}
\]

Using this we have:

\[
(1 + x + x^2) (1 - x + x^2) (1 - x^2 + x^4) \ldots \left(1 - x^{2n} + x^{2n+1}\right) = 1 + x^{2n+1} + x^{2n+2}.
\]

and rearranging gives:

\[
\frac{1}{1 + x + x^2} = (1 - x + x^2) (1 - x^2 + x^4) \ldots \left(1 - x^{2n} + x^{2n+1}\right) - \frac{x^{2n+1} + x^{2n+2}}{1 + x + x^2}
\]

Since \(|x| < 1\) as \(n \to \infty\) the last term tends to 0. We now have:

\[
\frac{1}{1 + x + x^2} = \prod_{r=0}^{\infty} \left(1 - x^{2r} + x^{2r+1}\right)
\]

Taking logs of both sides results in:

\[
-\ln(1 + x + x^2) = \sum_{r=0}^{\infty} \ln\left(1 - x^{2r} + x^{2r+1}\right)
\]

\[
\ln(1 + x + x^2) = -\sum_{r=0}^{\infty} \ln\left(1 - x^{2r} + x^{2r+1}\right)
\]

\[
\ln(1 + x + x^2) = -\ln(1 - x + x^2) - \ln(1 - x^2 + x^4) - \ln(1 - x^4 + x^8) \\
- \ln(1 - x^8 + x^{16}) - \ldots
\]

Then differentiating both sides gives:

\[
\frac{1 + 2x}{(1 + x + x^2)} = -\frac{1 + 2x}{(1 - x + x^2)} - \frac{-2x + 4x^3}{(1 - x^2 + x^4)} - \frac{-4x^3 + 8x^7}{(1 - x^4 + x^8)} - \ldots
\]

\[
= \frac{1 + 2x}{(1 - x + x^2)} + \frac{2x - 4x^3}{(1 - x^2 + x^4)} + \frac{4x^3 - 8x^7}{(1 - x^4 + x^8)} + \ldots
\]

Alternatively, you could replace \(x\) by \(x^3\) in the result \(\ln(1 - x) = -\sum_{r=0}^{\infty} \ln(1 + x^{2r})\) from part (i), then use the difference and sum of two cubes formulae, \(a^3 - b^3 = (a - b)(a^2 + ab + b^2)\) and \(a^3 + b^3 = (a + b)(a^2 - ab + b^2)\), and subtract the part (i) result before differentiating. Both ways are fine!
(i) If $\alpha$ is a root of both equations then we have:

\[ \alpha^2 + a\alpha + b = 0 \quad \text{and} \quad \alpha^2 + c\alpha + d = 0 \]

Evaluating (1) – (2) gives:

\[ \alpha(a - c) + (b - d) = 0 \]
\[ \alpha(a - c) = -(b - d) \]
\[ \alpha = \frac{b - d}{a - c} \quad \text{as} \quad a - c \neq 0 \]

Starting with the “if” part we have:

\[(b - d)^2 - a(b - d)(a - c) + b(a - c)^2 = 0 \quad \text{divide by} \quad (a - c)^2 \]
\[ \left(\frac{b - d}{a - c}\right)^2 - a\left(\frac{b - d}{a - c}\right) + b = 0 \]

and so \( x = \frac{b - d}{a - c} \) is a solution of \( x^2 + ax + b = 0 \). Substituting into the other equation gives:

\[ \begin{align*}
x^2 + cx + d &= \left(\frac{b - d}{a - c}\right)^2 + c\left(\frac{b - d}{a - c}\right) + d \\
&= \left(\frac{b - d}{a - c}\right)^2 + (c - a)\left(\frac{b - d}{a - c}\right) + a\left(\frac{b - d}{a - c}\right) + (d - b) + b \\
&= \left[\left(\frac{b - d}{a - c}\right)^2 + a\left(\frac{b - d}{a - c}\right) + b\right] + (c - a)\left(\frac{b - d}{a - c}\right) + (d - b) \\
&= [0] + (a - c)\frac{b - d}{a - c} + (d - b) \\
&= 0
\end{align*} \]

So \( x = \frac{b - d}{a - c} \) is a solution of both equations.

Going the other way (“Only if”), if the equations have a common root, \( \alpha \) then we have \( \alpha = \frac{b - d}{a - c} \). Substituting into \( x^2 + ax + b = 0 \) gives:

\[ \left(\frac{b - d}{a - c}\right)^2 + a\left(\frac{b - d}{a - c}\right) + b = 0 \quad \text{multiply by} \quad (a - c)^2 \]
\[ (b - d)^2 - a(b - d)(a - c) + b(a - c)^2 = 0 \]

Hence the equations have at least one common root if and only if \( (b - d)^2 - a(b - d)(a - c) + b(a - c)^2 = 0 \).

If we have \( (b - d)^2 - a(b - d)(a - c) + b(a - c)^2 = 0 \) and \( a = c \) then this implies that \( b - d = 0 \) and hence \( b = d \) and the two equations are the same (therefore must have at least one common root!).

If we have at least one common root, \( \alpha \), and \( a = c \), then we have \( \alpha^2 + a\alpha + b = 0 \) and \( \alpha^2 + a\alpha + d = 0 \) which implies that \( b = d \) and hence \( (b - d)^2 - a(b - d)(a - c) + b(a - c)^2 = 0 \) and so the result still holds when \( a = c \).
(ii) If

\[(b - r)^2 - a(b - r)(a + b - q) + b(a + b - q)^2 = 0\]

then letting \(d \to r\) and \(c \to q - b\) from part (i) shows that

\[x^2 + ax + b = 0\]  \hspace{1cm} \text{and} \hspace{1cm} (3)
\[x^2 + (q - b)x + r = 0\]  \hspace{1cm} \text{have a common root.} \hspace{1cm} (4)

If \(\alpha\) is a common root to (3) and (4), then it is also a root of \(x \times (3) + (4)\), i.e.:

\[x (x^2 + ax + b) + x^2 + (q - b)x + r = 0\]
\[x^3 + ax^2 + bx + x^2 + qx - bx + r = 0\]
\[x^3 + (a + 1)x^2 + qx + r = 0\]

Therefore, if \((b - r)^2 - a(b - r)(a + b - q) + b(a + b - q)^2 = 0\) then \(x^2 + ax + b = 0\) and \(x^3 + (a + 1)x^2 + cx + d = 0\) have at least one common root.

If \(x^2 + ax + b = 0\) and \(x^3 + (a + 1)x^2 + qx + r = 0\) have a common root \(\alpha\) then we have:

\[\alpha^2 + a\alpha + b = 0\]  \hspace{1cm} (5)
\[\alpha^3 + (a + 1)\alpha^2 + q\alpha + r = 0\]  \hspace{1cm} (6)

Then considering (6) - \(\alpha(5)\) we have:

\[\alpha^3 + (a + 1)\alpha^2 + q\alpha + r - \alpha (\alpha^2 + a\alpha + b) = 0\]
\[\alpha^3 + (a + 1)\alpha^2 + q\alpha + r - \alpha^2 - a\alpha^2 - b\alpha = 0\]
\[\alpha^2 + (q - b)\alpha + r = 0\]

Hence the two equations \(x^2 + ax + b = 0\) and \(x^2 + (q - b)x + r = 0\) have at least one common root, and so by using the result in part (i) we have:

\[(b - d)^2 - a(b - d)(a - c) + b(a - c)^2 = 0\] \hspace{1cm} \text{let} \ c = q - b\ \text{and} \ d = r
\[(b - r)^2 - a(b - r)(a - q + b) + b(a - q + b)^2 = 0\] \hspace{1cm} \text{as required.}

For the last part, take \(a = \frac{5}{2}\), \(q = \frac{5}{2}\) and \(r = \frac{1}{2}\), and then we have:

\[(b - r)^2 - a(b - r)(a - q + b) + b(a - q + b)^2 = 0\]
\[(b - \frac{1}{2})^2 - \frac{5}{2}(b - \frac{1}{2}) (\frac{5}{2} - \frac{5}{2} + b) + b(\frac{5}{2} - \frac{5}{2} + b)^2 = 0\]
\[b^2 - b + \frac{1}{4} - \frac{5}{2}b (b - \frac{1}{2}) + b^3 = 0\]
\[b^3 + b^2 - b + \frac{1}{4} - \frac{5}{2}b^2 + \frac{5}{4}b = 0\]
\[b^3 - \frac{5}{2}b^2 + \frac{1}{4}b + \frac{1}{4} = 0\]
\[4b^3 - 6b^2 + b + 1 = 0\]

One of the solutions is \(b = 1\), so the others are given by \(4b^2 - 2b - 1 = 0\), i.e.
\[b = \frac{2 \pm \sqrt{20}}{8} = \frac{1 \pm \sqrt{5}}{4}.\]