STEP Support Programme

STEP 3 Algebra Topic Notes

Summation of series

There are some formulae for series (including *Maclaurin series*) that you are expected to know for STEP. You can find a list of the required formulae at the back of the STEP specifications available on the Cambridge Assessment Admissions Testing website.

Partial fractions can often be used to rewrite a sum so that lot of terms cancel (the *method of differences*), leaving you with the first few and last few. By considering the sum of the first n terms can enable you to determine whether the sum converges or not (and if it does converge, the sum to infinity will exist).

Try finding $\displaystyle\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)}$ and then finding $\displaystyle\sum_{r=1}^{\infty} \frac{r+4}{r(r+1)(r+2)}$.

Polynomial equations

If a polynomial has real coefficients then the roots are real, or they occur in complex conjugate pairs[1]. There are always n roots of a polynomial of degree n (but some or all might be complex, and there might be repeated roots). This is known as the *Fundamental Theorem of Algebra*. A consequence of these two is that a cubic with real roots must always have at least one real root.

A cubic equation $ax^3 + bx^2 + cx + d = 0$ with roots α, β and γ satisfies:

$$\begin{aligned}\sum(\alpha) = \alpha + \beta + \gamma &= -\frac{b}{a}\\ \sum(\alpha\beta) = \alpha\beta + \beta\gamma + \gamma\alpha &= \frac{c}{a}\\ \alpha\beta\gamma &= -\frac{d}{a}\end{aligned}$$

You can derive these by equating coefficients from $ax^3 + bx^2 + cx + d \equiv a(x-\alpha)(x-\beta)(x-\gamma)$.

For a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ we have:

$$\begin{aligned}\sum(\alpha) = \alpha + \beta + \gamma + \delta &= -\frac{b}{a}\\ \sum(\alpha\beta) = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= \frac{c}{a}\\ \sum(\alpha\beta\gamma) = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta &= -\frac{d}{a}\\ \alpha\beta\gamma\delta &= \frac{e}{a}\end{aligned}$$

[1] It is important to note that if the polynomial has one or more complex coefficients then this does not hold.

You can use these results to find a new equation whose roots are related to the roots of the old equation. For example, if you wanted to find the cubic equation with roots $\alpha+1$, $\beta+1$, $\gamma+1$ you could use:

$$\begin{aligned}
\sum(\alpha') &= \sum(\alpha+1) = \alpha+\beta+\gamma+3 \\
\sum(\alpha'\beta') &= \sum\Big((\alpha+1)(\beta+1)\Big) = (\alpha+1)(\beta+1)+ \\
&\qquad\qquad\qquad\qquad\qquad (\beta+1)(\gamma+1)+(\gamma+1)(\alpha+1) \\
&\qquad\qquad\qquad\qquad = \alpha\beta+\beta\gamma+\gamma\alpha+2\,(\alpha+\beta+\gamma)+3 \\
\alpha'\beta'\gamma' &= (\alpha+1)(\beta+1)(\gamma+1) = \alpha\beta\gamma+(\alpha\beta+\beta\gamma+\gamma\alpha)+(\alpha+\beta+\gamma)+1
\end{aligned}$$

Probably an easier way to do this is to substitute $y = x+1$ into ax^3+bx^2+cx+d. However this substitution method does not work in all cases, for example if you want the cubic with roots $\dfrac{\alpha\beta}{\gamma}$, $\dfrac{\beta\gamma}{\alpha}$ and $\dfrac{\gamma\alpha}{\beta}$ then you would need to consider the sums and products of roots formulae.

Trigonometry

In the old STEP specifications, the formulae for $\sin A \sin B$ and $\sin A + \sin B$ (etc.) were explicitly mentioned. The formulae here might be helpful when solving old STEP questions, but under the current specifications you would be asked to derive the relevant formula(e) or be given it in the question.

$$\begin{aligned}
\sin A + \sin B &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\
\sin A - \sin B &= 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \\
\cos A + \cos B &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\
\cos A - \cos B &= -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)
\end{aligned}$$

For the last one, if we take $C = \frac{A+B}{2}$ and $D = \frac{A-B}{2}$, which means that $A = C+D$ and $B = C-D$, we get:

$$2\sin C \sin D = \cos(C-D) - \cos(C+D)\,.$$

This (and similar identities) might be useful in summing series.