STEP Support Programme

STEP 3 Calculus Questions

1 **2003 S3 Q1**

Given that $x+a>0$ and $x+b>0$, and that $b>a$, show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(\frac{x+a}{x+b}\right)=\frac{\sqrt{b-a}}{(x+b)\sqrt{a+b+2x}}$$

and find $\dfrac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{arcosh}\left(\dfrac{x+b}{x+a}\right)$.

Hence, or otherwise, integrate, for $x>-1$,

(i) $\displaystyle\int \frac{1}{(x+1)\sqrt{x+3}}\,\mathrm{d}x$,

(ii) $\displaystyle\int \frac{1}{(x+3)\sqrt{x+1}}\,\mathrm{d}x$.

[You may use the results $\dfrac{\mathrm{d}}{\mathrm{d}x}\arcsin x=\dfrac{1}{\sqrt{1-x^2}}$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{arcosh}\,x=\dfrac{1}{\sqrt{x^2-1}}$.]

2 **2010 S3 Q2**

In this question, a is a positive constant.

(i) Express $\cosh a$ in terms of exponentials.

By using partial fractions, prove that

$$\int_0^1 \frac{1}{x^2+2x\cosh a+1}\,\mathrm{d}x=\frac{a}{2\sinh a}\,.$$

(ii) Find, expressing your answers in terms of hyperbolic functions,

$$\int_1^\infty \frac{1}{x^2+2x\sinh a-1}\,\mathrm{d}x$$

and

$$\int_0^\infty \frac{1}{x^4+2x^2\cosh a+1}\,\mathrm{d}x\,.$$

3 **2002 S3 Q1**

Find the area of the region between the curve $y = \dfrac{\ln x}{x}$ and the x-axis, for $1 \leqslant x \leqslant a$. What happens to this area as a tends to infinity?

Find the volume of the solid obtained when the region between the curve $y = \dfrac{\ln x}{x}$ and the x-axis, for $1 \leqslant x \leqslant a$, is rotated through 2π radians about the x-axis. What happens to this volume as a tends to infinity?

4 **2004 S3 Q7**

For $n = 1, 2, 3, \ldots$, let

$$I_n = \int_0^1 \frac{t^{n-1}}{(t+1)^n}\,\mathrm{d}t\,.$$

By considering the greatest value taken by $\dfrac{t}{t+1}$ for $0 \leqslant t \leqslant 1$ show that $I_{n+1} < \frac{1}{2} I_n$.

Show also that $I_{n+1} = -\dfrac{1}{n\,2^n} + I_n$.

Deduce that $I_n < \dfrac{1}{n\,2^{n-1}}$.

Prove that

$$\ln 2 = \sum_{r=1}^{n} \frac{1}{r\,2^r} + I_{n+1}$$

and hence show that $\frac{2}{3} < \ln 2 < \frac{17}{24}$.