STEP Support Programme

STEP 3 Calculus Questions

1 2003 S3 Q1
Given that \( x + a > 0 \) and \( x + b > 0 \), and that \( b > a \), show that

\[
\frac{d}{dx} \arcsin \left( \frac{x + a}{x + b} \right) = \frac{\sqrt{b - a}}{(x + b) \sqrt{a + b + 2x}}
\]

and find \( \frac{d}{dx} \text{arcosh} \left( \frac{x + b}{x + a} \right) \).

Hence, or otherwise, integrate, for \( x > -1 \),

(i) \( \int \frac{1}{(x + 1) \sqrt{x + 3}} \, dx \),

(ii) \( \int \frac{1}{(x + 3) \sqrt{x + 1}} \, dx \).

[You may use the results \( \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}} \) and \( \frac{d}{dx} \text{arcosh} x = \frac{1}{\sqrt{x^2 - 1}} \).]

2 2010 S3 Q2
In this question, \( a \) is a positive constant.

(i) Express \( \cosh a \) in terms of exponentials.

By using partial fractions, prove that

\[
\int_0^1 \frac{1}{x^2 + 2x \cosh a + 1} \, dx = \frac{a}{2 \sinh a}.
\]

(ii) Find, expressing your answers in terms of hyperbolic functions,

\[
\int_1^\infty \frac{1}{x^2 + 2x \sinh a - 1} \, dx
\]

and

\[
\int_0^\infty \frac{1}{x^4 + 2x^2 \cosh a + 1} \, dx.
\]
3  **2002 S3 Q1**

Find the area of the region between the curve \( y = \frac{\ln x}{x} \) and the \( x \)-axis, for \( 1 \leq x \leq a \). What happens to this area as \( a \) tends to infinity?

Find the volume of the solid obtained when the region between the curve \( y = \frac{\ln x}{x} \) and the \( x \)-axis, for \( 1 \leq x \leq a \), is rotated through \( 2\pi \) radians about the \( x \)-axis. What happens to this volume as \( a \) tends to infinity?

4  **2004 S3 Q7**

For \( n = 1, 2, 3, \ldots \), let

\[
I_n = \int_0^1 \frac{t^{n-1}}{(t+1)n} \, dt.
\]

By considering the greatest value taken by \( \frac{t}{t+1} \) for \( 0 \leq t \leq 1 \) show that \( I_{n+1} < \frac{1}{2} I_n \).

Show also that \( I_{n+1} = -\frac{1}{n 2^n} + I_n \).

Deduce that \( I_n < \frac{1}{n 2^{n-1}} \).

Prove that

\[
\ln 2 = \sum_{r=1}^{n} \frac{1}{r 2^r} + I_{n+1}
\]

and hence show that \( \frac{2}{3} < \ln 2 < \frac{17}{24} \).