STEP Support Programme

STEP 3 Calculus: Solutions

1 We are given two useful formulae at the bottom of the question. Note that $\arcsin x = \sin^{-1} x$.

For the first result we have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(\frac{x+a}{x+b}\right) &= \frac{1}{\sqrt{1-\left(\frac{x+a}{x+b}\right)^2}} \times \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x+a}{x+b}\right)\\
&= \frac{x+b}{\sqrt{(x+b)^2-(x+a)^2}} \times \frac{(x+b)-(x+a)}{(x+b)^2}\\
&= \frac{1}{\sqrt{\left[(x+b)+(x+a)\right]\left[(x+b)-(x+a)\right]}} \times \frac{b-a}{x+b}\\
&= \frac{1}{\sqrt{(2x+a+b)(b-a)}} \times \frac{b-a}{x+b}\\
&= \frac{\sqrt{b-a}}{(x+b)\sqrt{a+b+2x}} \qquad \text{as required}
\end{aligned}$$

Note that the difference of two squares formula — $A^2 - B^2 = (A+B)(A-B)$ — has been used.

Similarly:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcosh}\left(\frac{x+b}{x+a}\right) &= \frac{1}{\sqrt{\left(\frac{x+b}{x+a}\right)^2-1}} \times \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x+b}{x+a}\right)\\
&= \frac{x+a}{\sqrt{(x+b)^2-(x+a)^2}} \times \frac{(x+a)-(x+b)}{(x+a)^2}\\
&= \frac{1}{\sqrt{\left[(x+b)+(x+a)\right]\left[(x+b)-(x+a)\right]}} \times \frac{a-b}{x+a}\\
&= \frac{1}{\sqrt{(a+b+2x)(b-a)}} \times \frac{a-b}{x+a}\\
&= -\frac{\sqrt{b-a}}{(x+a)\sqrt{a+b+2x}}
\end{aligned}$$

For the two requested integrals start by looking at them and considering the similarities between these and the derivatives found in the stem of the question. Looking at them suggests that $a = 1$, $b = 3$ might be a good thing to consider (remember that $b > a$).

(i) Using $a = 1, b = 3$ in the second result found in the stem gives:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcosh}\left(\frac{x+b}{x+a}\right) &= -\frac{\sqrt{b-a}}{(x+a)\sqrt{a+b+2x}}\\
\implies \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcosh}\left(\frac{x+3}{x+1}\right) &= -\frac{\sqrt{3-1}}{(x+1)\sqrt{1+3+2x}}\\
&= -\frac{1}{(x+1)\sqrt{2+x}}
\end{aligned}$$

This is not quite what we wanted, so try $a = 1, b = 5$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcosh}\left(\frac{x+5}{x+1}\right) &= -\frac{\sqrt{5-1}}{(x+1)\sqrt{1+5+2x}} \\
&= -\frac{2}{(x+1)\sqrt{6+2x}} \\
\implies \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcosh}\left(\frac{x+5}{x+1}\right) &= -\frac{\sqrt{2}}{(x+1)\sqrt{3+x}}
\end{aligned}$$

Therefore we have:

$$\int \frac{1}{(x+1)\sqrt{x+3}}\,\mathrm{d}x = -\frac{1}{\sqrt{2}}\operatorname{arcosh}\left(\frac{x+5}{x+1}\right) + c$$

(ii) Since part **(i)** used the second result from the stem, it is probable that this part will need the first result. Take $b = 3$ and $a = -1$ to get:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(\frac{x+a}{x+b}\right) &= \frac{\sqrt{b-a}}{(x+b)\sqrt{a+b+2x}} \\
\implies \frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(\frac{x-1}{x+3}\right) &= \frac{\sqrt{3-(-1)}}{(x+3)\sqrt{-1+3+2x}} \\
\implies \frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(\frac{x-1}{x+3}\right) &= \frac{\sqrt{2}}{(x+3)\sqrt{x+1}}
\end{aligned}$$

Therefore we have:

$$\int \frac{1}{(x+3)\sqrt{x+1}}\,\mathrm{d}x = \frac{1}{\sqrt{2}}\arcsin\left(\frac{x-1}{x+3}\right) + c$$

2 **(i)** $\cosh a = \frac{1}{2}\left(\mathrm{e}^{a} + \mathrm{e}^{-a}\right)$. This gives us:

$$\begin{aligned}\int_0^1 \frac{1}{x^2 + 2x\cosh a + 1}\,\mathrm{d}x &= \int_0^1 \frac{1}{x^2 + x\left(\mathrm{e}^{a} + \mathrm{e}^{-a}\right) + 1}\,\mathrm{d}x \\ &= \int_0^1 \frac{1}{\left(x + \mathrm{e}^{a}\right)\left(x + \mathrm{e}^{-a}\right)}\,\mathrm{d}x\end{aligned}$$

To use partial fractions we want to find A and B such that:

$$\frac{A}{x + \mathrm{e}^{a}} + \frac{B}{x + \mathrm{e}^{-a}} \equiv \frac{1}{\left(x + \mathrm{e}^{a}\right)\left(x + \mathrm{e}^{-a}\right)} \qquad \text{i.e.}$$

$$A\left(x + \mathrm{e}^{-a}\right) + B\left(x + \mathrm{e}^{a}\right) \equiv 1$$

Equating coefficients of x gives $A = -B$, and equating constants gives $B = \dfrac{1}{\mathrm{e}^{a} - \mathrm{e}^{-a}} = \dfrac{1}{2\sinh a}$. The integral becomes:

$$\begin{aligned}\frac{1}{2\sinh a}\int_0^1 \left(\frac{1}{x + \mathrm{e}^{-a}}\right) - \left(\frac{1}{x + \mathrm{e}^{a}}\right)\,\mathrm{d}x &= \frac{1}{2\sinh a}\Big[\ln\left(x + \mathrm{e}^{-a}\right) - \ln\left(x + \mathrm{e}^{a}\right)\Big]_0^1 \\ &= \frac{1}{2\sinh a}\left[\ln\left(\frac{x + \mathrm{e}^{-a}}{x + \mathrm{e}^{a}}\right)\right]_0^1 \\ &= \frac{1}{2\sinh a}\left[\ln\left(\frac{1 + \mathrm{e}^{-a}}{1 + \mathrm{e}^{a}}\right) - \ln\left(\frac{\mathrm{e}^{-a}}{\mathrm{e}^{a}}\right)\right] \\ &= \frac{1}{2\sinh a}\left[\ln\left(\frac{\left(1 + \mathrm{e}^{-a}\right)\times\left(\mathrm{e}^{a}\right)}{\left(1 + \mathrm{e}^{a}\right)\times\left(\mathrm{e}^{-a}\right)}\right)\right] \\ &= \frac{1}{2\sinh a}\left[\ln\left(\mathrm{e}^{a}\times\cancel{\frac{1 + \mathrm{e}^{-a}}{\mathrm{e}^{-a} + 1}}\right)\right] \\ &= \frac{1}{2\sinh a}\ln\left(\mathrm{e}^{a}\right) \\ &= \frac{a}{2\sinh a} \qquad \text{as required.}\end{aligned}$$

(ii) Starting in a similar way to part **(i)**, we note that $\sinh a = \frac{1}{2}\left(\mathrm{e}^{a} - \mathrm{e}^{-a}\right)$. This gives:

$$\begin{aligned}\int_1^\infty \frac{1}{x^2 + 2x\sinh a - 1}\,\mathrm{d}x &= \int_1^\infty \frac{1}{x^2 + x\left(\mathrm{e}^{a} - \mathrm{e}^{-a}\right) - 1}\,\mathrm{d}x \\ &= \int_1^\infty \frac{1}{\left(x + \mathrm{e}^{a}\right)\left(x - \mathrm{e}^{-a}\right)}\,\mathrm{d}x\end{aligned}$$

Then using partial fractions gives:

$$\begin{aligned}
\int_1^\infty \frac{1}{(x+\mathrm{e}^a)(x-\mathrm{e}^{-a})}\,\mathrm{d}x &= \frac{1}{2\cosh a}\int_1^\infty \frac{1}{x-\mathrm{e}^{-a}} - \frac{1}{x+\mathrm{e}^a}\,\mathrm{d}x \\
&= \frac{1}{2\cosh a}\Big[\ln\left(x-\mathrm{e}^{-a}\right) - \ln\left(x+\mathrm{e}^a\right)\Big]_1^\infty \\
&= \frac{1}{2\cosh a}\left[\ln\left(\frac{x-\mathrm{e}^{-a}}{x+\mathrm{e}^a}\right)\right]_1^\infty \\
&= \frac{1}{2\cosh a}\left[0-\ln\left(\frac{1-\mathrm{e}^{-a}}{1+\mathrm{e}^a}\right)\right] \\
&= \frac{1}{2\cosh a}\left[\ln\left(\frac{1+\mathrm{e}^a}{1-\mathrm{e}^{-a}}\right)\right] \\
&= \frac{1}{2\cosh a}\left[\ln\left(\frac{(1+\mathrm{e}^a)\,\mathrm{e}^a}{\mathrm{e}^a-1}\right)\right] \\
&= \frac{1}{2\cosh a}\left[a+\ln\left(\frac{\mathrm{e}^a+1}{\mathrm{e}^a-1}\right)\right] \\
&= \frac{1}{2\cosh a}\left[a+\ln\left(\frac{\mathrm{e}^{a/2}+\mathrm{e}^{-a/2}}{\mathrm{e}^{a/2}-\mathrm{e}^{-a/2}}\right)\right] \\
&= \frac{1}{2\cosh a}\Big[a+\ln\Big(\coth\left(\tfrac{a}{2}\right)\Big)\Big] \\
&\qquad \text{or} \quad \frac{1}{2\cosh a}\Big[a-\ln\Big(\tanh\left(\tfrac{a}{2}\right)\Big)\Big]
\end{aligned}$$

For the last integral, start by using the same method as in part **(i)** again:

$$\begin{aligned}
\int_0^\infty \frac{1}{x^4+2x^2\cosh a+1}\,\mathrm{d}x &= \int_0^\infty \frac{1}{x^4+x^2\left(\mathrm{e}^a+\mathrm{e}^{-a}\right)+1}\,\mathrm{d}x \\
&= \int_0^\infty \frac{1}{(x^2+\mathrm{e}^a)(x^2+\mathrm{e}^{-a})}\,\mathrm{d}x \\
&= \frac{1}{2\sinh a}\int_0^\infty \frac{1}{x^2+\mathrm{e}^{-a}} - \frac{1}{x^2+\mathrm{e}^a}\,\mathrm{d}x
\end{aligned}$$

Then, using the integral $\displaystyle\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$[1] we have:

$$\begin{aligned}
\frac{1}{2\sinh a}\int_0^{\infty} \frac{1}{x^2+\mathrm{e}^{-a}} - \frac{1}{x^2+\mathrm{e}^{a}}\,\mathrm{d}x &= \frac{1}{2\sinh a}\left[\frac{1}{\sqrt{\mathrm{e}^{-a}}}\tan^{-1}\left(\frac{x}{\sqrt{\mathrm{e}^{-a}}}\right) - \frac{1}{\sqrt{\mathrm{e}^{a}}}\tan^{-1}\left(\frac{x}{\sqrt{\mathrm{e}^{a}}}\right)\right]_0^{\infty} \\
&= \frac{1}{2\sinh a}\left[\mathrm{e}^{a/2}\tan^{-1}\left(x\mathrm{e}^{a/2}\right) - \mathrm{e}^{-a/2}\tan^{-1}\left(x\mathrm{e}^{-a/2}\right)\right]_0^{\infty} \\
&= \frac{1}{2\sinh a}\left[\mathrm{e}^{a/2}\times\frac{\pi}{2} - \mathrm{e}^{-a/2}\times\frac{\pi}{2} - 0\right] \\
&= \frac{\pi}{2}\left(\frac{\mathrm{e}^{a/2}-\mathrm{e}^{-a/2}}{2\sinh a}\right) \\
&= \frac{\pi}{2}\left(\frac{\mathrm{e}^{a/2}-\mathrm{e}^{-a/2}}{\mathrm{e}^{a}-\mathrm{e}^{-a}}\right) \\
&= \frac{\pi}{2}\left(\frac{\mathrm{e}^{a/2}-\mathrm{e}^{-a/2}}{\left(\mathrm{e}^{a/2}-\mathrm{e}^{-a/2}\right)\left(\mathrm{e}^{a/2}+\mathrm{e}^{-a/2}\right)}\right) \\
&= \frac{\pi}{2}\left(\frac{1}{\mathrm{e}^{a/2}+\mathrm{e}^{-a/2}}\right) \\
&= \frac{\pi}{4\cosh\frac{a}{2}}
\end{aligned}$$

[1] When this question was set this formula was in the accompanying formula book. If this question was set now this formula might be given in the question (and since there is a lot to do in this question, I expect that the formula would be given).

3 This looks as if integration by parts might be useful. Let $I = \int \frac{\ln x}{x}\,\mathrm{d}x$. We then have:

$$\begin{aligned} I &= \int \ln x \times \frac{1}{x}\,\mathrm{d}x \\ &= \Big[\ln x \times \ln x\Big] - \int \frac{1}{x} \times \ln x\,\mathrm{d}x \\ &= (\ln x)^2 - I \end{aligned}$$

Hence we have $I = \frac{1}{2}(\ln x)^2 + c$ and:

$$\int_1^a \frac{\ln x}{x}\,\mathrm{d}x = \frac{1}{2}(\ln a)^2.$$

As $a \to \infty$, $\frac{1}{2}(\ln a)^2 \to \infty$, so the area tends to infinity as a tends to infinity.

For the volume we have:

$$\begin{aligned} V &= \pi \int_1^a \left(\frac{\ln x}{x}\right)^2\,\mathrm{d}x \\ &= \pi\Big[(\ln x)^2 \times -x^{-1}\Big]_1^a + \pi \int_1^a 2(\ln x)\frac{1}{x} \times x^{-1}\,\mathrm{d}x \\ &= \pi(\ln a)^2 \times -a^{-1} + \pi \int_1^a 2(\ln x) \times x^{-2}\,\mathrm{d}x \\ &= -\frac{\pi(\ln a)^2}{a} + 2\pi\left(\Big[\ln x \times -x^{-1}\Big]_1^a + \int_1^a \frac{1}{x} \times x^{-1}\,\mathrm{d}x\right) \\ &= -\frac{\pi(\ln a)^2}{a} - 2\pi\frac{\ln a}{a} + 2\pi\Big[-x^{-1}\Big]_1^a \\ &= -\frac{\pi(\ln a)^2}{a} - 2\pi\frac{\ln a}{a} - 2\pi\frac{1}{a} + 2\pi \\ &= \pi\left(2 - \frac{(\ln a)^2}{a} - \frac{2\ln a}{a} - \frac{2}{a}\right) \end{aligned}$$

Then as $a \to \infty$ we have $\frac{\ln a}{a} \to 0$ and $\frac{(\ln a)^2}{a} \to 0$ (and also $\frac{2}{a} \to 0$). Hence the volume tends to 2π.

- Note that the area under the curve is infinite, but that the volume obtained by rotating around is finite. This means that the shape could not be painted, but could be filled.

- This seems rather short for a STEP question by today's standards. Also, the result that $\frac{\ln a}{a} \to 0$ as $a \to \infty$ is not completely obvious, and I would expect a modern day STEP question to either give you this result or ask you to prove it.

  Consider $\frac{\ln x}{x}$ and let $x \to \infty$. Now let $x = \mathrm{e}^y$, and so as $x \to \infty$ we have $y \to \infty$. So:

$$\begin{aligned} \lim_{x\to\infty} \frac{\ln x}{x} &= \lim_{y\to\infty} \frac{y}{\mathrm{e}^y} \\ &= \lim_{y\to\infty} \frac{y}{1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots} \\ &= \lim_{y\to\infty} \frac{1}{\frac{1}{y} + 1 + \frac{y}{2!} + \frac{y^2}{3!} + \cdots} \\ &= 0 \end{aligned}$$

A similar argument can be used to show that $\lim_{x\to\infty} \frac{(\ln x)^2}{x} = 0$, and can be extended to show that $\lim_{x\to\infty} \frac{(\ln x)^m}{x}$ for integer $m \geqslant 1$.

Another way of showing that $\lim_{x\to\infty} \frac{\ln x}{x} = 0$ is to use L'Hôpital's rule. You are not expected to know this for STEP!

4 Differentiating $\dfrac{t}{1+t}$ gives $\dfrac{(1+t)-t}{(1+t)^2} = \dfrac{1}{(1+t)^2}$ and so the function is increasing as t increases. Hence the maximum in the range $0 \leqslant t \leqslant 1$ is when $t = 1$ and has value $\frac{1}{2}$.

Note that care would have had to be taken if the range of t had included the vertical asymptote when $t = -1$.

Hence we have $\dfrac{t}{1+t} \leqslant \frac{1}{2}$.

Now consider I_{n+1}. We have:

$$\begin{aligned} I_{n+1} &= \int_0^1 \frac{t^n}{(1+t)^{n+1}}\,\mathrm{d}t \\ &= \int_0^1 \frac{t}{1+t} \times \frac{t^{n-1}}{(1+t)^n}\,\mathrm{d}t \\ &< \tfrac{1}{2}\int_0^1 \frac{t^{n-1}}{(1+t)^n}\,\mathrm{d}t = \tfrac{1}{2}I_n \end{aligned}$$

Integration by parts gives:

$$\begin{aligned} \int_0^1 \frac{t^n}{(1+t)^{n+1}}\,\mathrm{d}t &= \left[t^n \times -\tfrac{1}{n}(1+t)^{-n}\right]_0^1 + \int_0^1 nt^{n-1} \times \tfrac{1}{n}(1+t)^{-n}\,\mathrm{d}t \\ &= -\frac{1}{n2^n} + \int_0^1 \frac{t^{n-1}}{(1+t)^n}\,\mathrm{d}t \\ &= -\frac{1}{n2^n} + I_n \end{aligned}$$

Combining the two results we have:

$$\begin{aligned} -\frac{1}{n2^n} + I_n &= I_{n+1} \\ -\frac{1}{n2^n} + I_n &< \tfrac{1}{2}I_n \\ \tfrac{1}{2}I_n &< \frac{1}{n2^n} \\ I_n &< \frac{2}{n2^n} \\ I_n &< \frac{1}{n2^{n-1}} \end{aligned}$$

We have $I_{n+1} - I_n = -\dfrac{1}{n2^n}$, and so:

$$\begin{aligned} (I_{n+1} - I_n) + (I_n - I_{n-1}) + \cdots + (I_2 - I_1) &= -\sum_{r=1}^{n} \frac{1}{r2^r} \\ I_{n+1} - I_1 &= -\sum_{r=1}^{n} \frac{1}{r2^r} \\ I_{n+1} + \sum_{r=1}^{n} \frac{1}{r2^r} &= I_1 \end{aligned}$$

We also have:

$$\begin{aligned} I_1 &= \int_0^1 \frac{1}{t+1} \\ &= \Big[\ln(1+t)\Big]_0^1 \\ &= \ln 2 \end{aligned}$$

Hence $I_{n+1} + \displaystyle\sum_{r=1}^{n} \frac{1}{r2^r} = \ln 2$. This means that $\ln 2 = \displaystyle\sum_{r=1}^{n} \frac{1}{r2^r} +$ something positive.

For the inequalities first note that $\ln 2 > \displaystyle\sum_{r=1}^{n} \frac{1}{r2^r}$. In particular, taking $n = 3$ gives:

$$\begin{aligned} \ln 2 &> \sum_{r=1}^{3} \frac{1}{r2^r} \\ \ln 2 &> \frac{1}{2} + \frac{1}{8} + \frac{1}{24} \\ \ln 2 &> \frac{12}{24} + \frac{3}{24} + \frac{1}{24} \\ \ln 2 &> \frac{16}{24} \\ \ln 2 &> \frac{2}{3} \end{aligned}$$

We also have $I_n < \dfrac{1}{n2^{n-1}}$, and so:

$$\begin{aligned} \ln 2 &= \sum_{r=1}^{n} \frac{1}{r2^r} + I_{n+1} \\ \ln 2 &< \sum_{r=1}^{n} \frac{1}{r2^r} + \frac{1}{(n+1)2^n} \end{aligned}$$

Taking $n = 2$ gives:

$$\begin{aligned} \ln 2 &< \sum_{r=1}^{2} \frac{1}{r2^r} + \frac{1}{(3) \times 2^2} \\ \ln 2 &< \frac{1}{2} + \frac{1}{8} + \frac{1}{12} \\ \ln 2 &< \frac{12}{24} + \frac{3}{24} + \frac{2}{24} \\ \ln 2 &< \frac{17}{24} \end{aligned}$$

And so we have $\frac{2}{3} < \ln 2 < \frac{17}{24}$ as required.