STEP Support Programme

STEP 2 Calculus Questions

1  2005 S2 Q1
Find the three values of \( x \) for which the derivative of \( x^2e^{-x^2} \) is zero.

Given that \( a \) and \( b \) are distinct positive numbers, find a polynomial \( P(x) \) such that the derivative of \( P(x)e^{-x^2} \) is zero for \( x = 0, x = \pm a \) and \( x = \pm b \), but for no other values of \( x \).

2  2006 S2 Q4
By making the substitution \( x = \pi - t \), show that

\[
\int_0^\pi xf(x)dx = \frac{1}{2}\pi\int_0^\pi f(x)dx,
\]

where \( f(x) \) is a given function of \( x \).

Evaluate the following integrals:

(i) \( \int_0^\pi \frac{x\sin x}{3 + \sin^2 x} \, dx \);

(ii) \( \int_0^{2\pi} \frac{x\sin x}{3 + \sin^2 x} \, dx \);

(iii) \( \int_0^\pi \frac{x|\sin 2x|}{3 + \sin^2 x} \, dx \).

3  2007 S2 Q6
(i) Differentiate \( \ln(x + \sqrt{3 + x^2}) \) and \( x\sqrt{3 + x^2} \) and simplify your answers.

Hence find \( \int \sqrt{3 + x^2} \, dx \).

(ii) Find the two solutions of the differential equation

\[
3\left( \frac{dy}{dx} \right)^2 + 2x \frac{dy}{dx} = 1
\]

that satisfy \( y = 0 \) when \( x = 1 \).
4 2008 S2 Q7

(i) By writing \( y = u(1 + x^2)^{\frac{1}{2}} \), where \( u \) is a function of \( x \), find the solution of the equation

\[
\frac{1}{y} \frac{dy}{dx} = xy + \frac{x}{1 + x^2}
\]

for which \( y = 1 \) when \( x = 0 \).

(ii) Find the solution of the equation

\[
\frac{1}{y} \frac{dy}{dx} = x^2y + \frac{x^2}{1 + x^3}
\]

for which \( y = 1 \) when \( x = 0 \).

(iii) Give, without proof, a conjecture for the solution of the equation

\[
\frac{1}{y} \frac{dy}{dx} = x^{n-1}y + \frac{x^{n-1}}{1 + x^n}
\]

for which \( y = 1 \) when \( x = 0 \), where \( n \) is an integer greater than 1.

If you are finding question 4 quite tricky you might like to tackle the question below first (which uses the same ideas but is a STEP I question).

5 2010 S1 Q6

Show that, if \( y = e^x \), then

\[
(x - 1) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 0. \tag{*}
\]

In order to find other solutions of this differential equation, now let \( y = ue^x \), where \( u \) is a function of \( x \). By substituting this into (\(*\)), show that

\[
(x - 1) \frac{d^2u}{dx^2} + (x - 2) \frac{du}{dx} = 0. \tag{**}
\]

By setting \( \frac{du}{dx} = v \) in (***) and solving the resulting first order differential equation for \( v \), find \( u \) in terms of \( x \). Hence show that \( y = Ax + Be^x \) satisfies (\(*\)), where \( A \) and \( B \) are any constants.