These are not fully worked solutions — you need to fill in the gaps. It is a good idea to look at the
“Hints” document before this one.

1 For the first part, \( x = -1, 0, 1. \)

We need \( P'(x) - 2xP(x) \equiv k(x - a)(x + a)(x - b)(x + b), \) though we actually could have \( P'(x) - 2xP(x) \) having factors of \((x - a)^2\) etc., but we might as well use the simplest case!

This means that \( P(x) \) must be a quartic, so we can write \( P(x) = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon. \)

You can then use \( P'(x) - 2xP(x) \equiv x(x^2 - a^2)(x^2 - b^2) \) and equate coefficients to find the
values of the coefficients of \( P(x). \) Finding \( \alpha, \beta \) and \( \delta \) is fairly easy,
\( \gamma \) and \( \epsilon \) are a little harder.

\[
P(x) = -\frac{x^4}{2} + \left( \frac{a^2}{2} + \frac{b^2}{2} - 1 \right) x^2 + \left( \frac{a^2}{2} + \frac{b^2}{2} - \frac{a^2b^2}{2} - 1 \right).
\]

2 You should start by showing that

\[
I = \int_0^{\pi} x f(\sin x) \, dx = \int_0^{\pi} (\pi - t) f(\sin t) \, dt.
\]

You can then show that \( 2I = \int_0^{\pi} (x + \pi - x) f(\sin x) \, dx. \)

You should expect to use this repeatedly in the rest of the question!

(i) If you use \( u = \cos x \) then you should find that the integral is equal to
\( \frac{1}{2} \pi \int_{-1}^{1} \frac{1}{4 - u^2} \, du. \)

Using partial fractions we have \( \frac{1}{4 - u^2} = \frac{A}{2 + u} + \frac{B}{2 - u} \) and then you can find \( A \) and \( B. \)

The final answer is \( \frac{1}{4} \pi \ln 3. \)

(ii) Split the integral into two parts, and then use a substitution of \( t = x - \pi, \) which will
give you:

\[
\int_0^{\pi} \frac{x \sin x}{3 + \sin^2 x} \, dx + \int_0^{\pi} \frac{(t + \pi) \sin(t + \pi)}{3 + \sin^2(t + \pi)} \, dt.
\]

You can then use the fact that \( \sin(t+\pi) = -\sin t \) to reduce the integrals to
\(-\pi \int_0^{\pi} \frac{\sin t}{3 + \sin^2 t} \)
and, by using your work from part (i) the answer is \( -\frac{1}{2} \pi \ln 3. \)

(iii) Using \( \sin 2x = 2 \sin x \cos x, \) the stem result and a substitution \( u = \cos x \) leads to the
integral \( \frac{1}{2} \pi \int_{-1}^{1} \frac{2|u|}{4 - u^2} \, du \) which is the same as
\( \pi \int_{0}^{1} \frac{2u}{4 - u^2} \, du. \)

The result \( \int \frac{f'(x)}{f(x)} = \ln |f(x)| + c \) may be useful. Final answer \( \pi \ln \frac{4}{3}. \)
3  (i) The first two answers should be \( \frac{1}{\sqrt{3 + x^2}} \) and \( \frac{2x^2 + 3}{\sqrt{3 + x^2}} \).

Then we can write
\[
\sqrt{3 + x^2} = \frac{1}{2} \left( \frac{3 + 2x^2}{\sqrt{3 + x^2}} + \frac{3}{\sqrt{3 + x^2}} \right)
\]
and hence use the previous two answers to find \( \int \sqrt{3 + x^2} \, dx \). Don’t forget the constant of integration!

Final answer: \( \frac{1}{2} x \sqrt{3 + x^2} + \frac{3}{2} \ln \left( x + \sqrt{3 + x^2} \right) + c. \)

(ii) We can treat this as a quadratic equation, which leads to two differential equations:
\[
\frac{dy}{dx} = -x \pm \frac{\sqrt{x^2 + 3}}{3}.
\]

Integrating and using the given initial condition then gives:
\[
y_1 = -\frac{1}{6} x^2 + \frac{1}{6} x \sqrt{3 + x^2} + \frac{1}{2} \ln \left( x + \sqrt{3 + x^2} \right) - \frac{1}{6} - \frac{1}{2} \ln 3
\]
and
\[
y_2 = -\frac{1}{6} x^2 - \frac{1}{6} x \sqrt{3 + x^2} - \frac{1}{2} \ln \left( x + \sqrt{3 + x^2} \right) + \frac{1}{2} + \frac{1}{2} \ln 3.
\]
4  (i) With the given substitution we have \( \frac{dy}{dx} = (1 + x^2)^{\frac{1}{2}} \times \frac{du}{dx} + xu (1 + x^2)^{-\frac{1}{2}}. \)

This should reduce the differential equation to one where the variables are separable. The integral in \( x \) can be tackled with a substitution, or by inspection (“guessing” what the answer is and then checking it). You should find that \(-\frac{1}{u} = \frac{1}{3} (1 + x^2)^{\frac{3}{2}} + c.\)

You do need to give the final answer in terms of \( y \), and use the initial condition. The final answer is:

\[
y = \frac{3 (1 + x^2)^{\frac{1}{2}}}{4 - (1 + x^2)^{\frac{3}{2}}}.
\]

(ii) This time use \( y = u(1 + x^3)^{\frac{1}{3}} \). The final answer is:

\[
y = \frac{4 (1 + x^3)^{\frac{1}{3}}}{5 - (1 + x^3)^{\frac{4}{3}}}.
\]

(iii) This part requires you to look at the cases \( n = 2 \) (part (i)) and \( n = 3 \) (part (iii)) and generalise in terms of \( n \). Final answer:

\[
y = \frac{(n + 1) (1 + x^n)^{\frac{1}{n}}}{(n + 2) - (1 + x^n)^{\frac{n+1}{n}}}.
\]

Note: in each case equivalent forms would be acceptable, but it is easier to find the general case if you simplify the \( n = 2 \) and \( n = 3 \) cases as shown.
If you substitute \( y = e^x \) into equation (\( * \)) you get:

\[
(x - 1) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = (x - 1)e^x - xe^x + e^x = 0
\]

and so \( y = e^x \) is a solution of (\( * \)).

When substituting \( y = ue^x \), remember that \( u \) is a function of \( x \) so we have

\[
\frac{dy}{dx} = ue^x + \frac{du}{dx}e^x
\]

and

\[
\frac{d^2y}{dx^2} = \left( ue^x + \frac{du}{dx}e^x \right) + \left( \frac{d^u}{dx}e^x + \frac{d^2u}{dx^2}e^x \right).
\]

You can now substitute these into (\( * \)), and you can divide throughout by \( e^x \) (this is OK as we have \( e^x \neq 0 \)). With some simplification you should obtain (\( ** \)).

Setting \( \frac{du}{dx} = v \) gives the equation

\[
\frac{1}{v} \frac{dv}{dx} = -\frac{x - 2}{x - 1}.
\]

By using partial fractions\(^1\) we get

\[
\frac{x - 2}{x - 1} = 1 - \frac{1}{x - 1}.
\]

Integrating gives:

\[
\ln |v| = -x + \ln |x - 1| + c
\]

and so:

\[
v = ke^{-x}(x - 1) \quad \text{where} \quad k = e^c.
\]

We now have \( \frac{du}{dv} = ke^{-x}(x - 1) \) and so we have:

\[
u = \int ke^{-x}(x - 1)dx
\]

and you can use integration by parts to obtain

\[
u = -kxe^{-x} + c'.
\]

The final instruction is a “Hence”, so you need to use the previous results. In particular setting \( y = Ax + Be^x \) and showing that this satisfies (\( * \)) will not gain you any credit. Since \( y = ue^x \) we have \( y = -kx + c'e^x \) and so if we let \( A = -k \) and \( B = c' \) we know that \( y = Ax + Be^x \) satisfies (\( * \)).

\(^1\) Or we can note that \( \frac{x - 2}{x - 1} = \frac{x - 1 - 1}{x - 1} \).