STEP Support Programme

STEP 2 Calculus Questions: Solutions

These are not fully worked solutions — you need to fill in the gaps. It is a good idea to look at the "Hints" document before this one.

1 For the first part, $x = -1, 0, 1$.

We need $\mathrm{P}'(x) - 2x\mathrm{P}(x) \equiv kx(x-a)(x+a)(x-b)(x+b)$, though we actually could have $\mathrm{P}'(x) - 2x\mathrm{P}(x)$ having factors of $(x-a)^2$ etc., but we might as well use the simplest case! This means that $\mathrm{P}(x)$ must be a quartic, so we can write $\mathrm{P}(x) = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon$.

You can then use $\mathrm{P}'(x) - 2x\mathrm{P}(x) \equiv x(x^2 - a^2)(x^2 - b^2)$ and equate coefficients to find the values of the coefficients of $\mathrm{P}(x)$. Finding α, β and δ is fairly easy, γ and ϵ are a little harder.

$$\mathrm{P}(x) = -\frac{x^4}{2} + \left(\frac{a^2}{2} + \frac{b^2}{2} - 1\right)x^2 + \left(\frac{a^2}{2} + \frac{b^2}{2} - \frac{a^2b^2}{2} - 1\right).$$

2 You should start by showing that

$$I = \int_0^\pi x\mathrm{f}(\sin x)\mathrm{d}x = \int_0^\pi (\pi - t)\mathrm{f}(\sin t)\mathrm{d}t\,.$$

You can then show that $2I = \int_0^\pi (x + \pi - x)\mathrm{f}(\sin x)\mathrm{d}x$.

You should expect to use this repeatedly in the rest of the question!

(i) If you use $u = \cos x$ then you should find that the integral is equal to $\frac{1}{2}\pi \displaystyle\int_{-1}^{1} \frac{1}{4-u^2}\mathrm{d}u$.

Using partial fractions we have $\dfrac{1}{4-u^2} = \dfrac{A}{2+u} + \dfrac{B}{2-u}$ and then you can find A and B.

The final answer is $\frac{1}{4}\pi \ln 3$.

(ii) Split the integral into two parts, and then use a substitution of $t = x - \pi$, which will give you:

$$\int_0^\pi \frac{x\sin x}{3+\sin^2 x}\mathrm{d}x + \int_0^\pi \frac{(t+\pi)\sin(t+\pi)}{3+\sin^2(t+\pi)}\mathrm{d}t\,.$$

You can then use the fact that $\sin(t+\pi) = -\sin t$ to reduce the integrals to $-\pi \displaystyle\int_0^\pi \frac{\sin t}{3+\sin^2 t}$ and, by using your work from part **(i)** the answer is $-\frac{1}{2}\pi\ln 3$.

(iii) Using $\sin 2x = 2\sin x\cos x$, the stem result and a substitution $u = \cos x$ leads to the integral $\frac{1}{2}\pi \displaystyle\int_{-1}^{1} \frac{2|u|}{4-u^2}\mathrm{d}u$ which is the same as $\pi \displaystyle\int_0^1 \frac{2u}{4-u^2}\mathrm{d}u$.

The result $\displaystyle\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} = \ln|\mathrm{f}(x)| + c$ may be useful. Final answer $\pi \ln \frac{4}{3}$.

3 **(i)** The first two answers should be $\dfrac{1}{\sqrt{3+x^2}}$ and $\dfrac{2x^2+3}{\sqrt{3+x^2}}$.

Then we can write

$$\sqrt{3+x^2} = \frac{1}{2}\left(\frac{3+2x^2}{\sqrt{3+x^2}} + \frac{3}{\sqrt{3+x^2}}\right)$$

and hence use the previous two answers to find $\displaystyle\int \sqrt{3+x^2}\,\mathrm{d}x$. Don't forget the constant of integration!

Final answer: $\frac{1}{2}x\sqrt{3+x^2} + \frac{3}{2}\ln\left(x+\sqrt{3+x^2}\right) + c$.

(ii) We can treat this as a quadratic equation, which leads to two differential equations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x \pm \sqrt{x^2+3}}{3}\,.$$

Integrating and using the given initial condition then gives:

$$y_1 = -\tfrac{1}{6}x^2 + \tfrac{1}{6}x\sqrt{3+x^2} + \tfrac{1}{2}\ln\left(x+\sqrt{3+x^2}\right) - \tfrac{1}{6} - \tfrac{1}{2}\ln 3$$

and

$$y_2 = -\tfrac{1}{6}x^2 - \tfrac{1}{6}x\sqrt{3+x^2} - \tfrac{1}{2}\ln\left(x+\sqrt{3+x^2}\right) + \tfrac{1}{2} + \tfrac{1}{2}\ln 3\,.$$

4 **(i)** With the given substitution we have $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \left(1+x^2\right)^{\frac{1}{2}} \times \dfrac{\mathrm{d}u}{\mathrm{d}x} + xu\left(1+x^2\right)^{-\frac{1}{2}}$.

This should reduce the differential equation to one where the variables are separable. The integral in x can be tackled with a substitution, or by inspection ("guessing" what the answer is and then checking it). You should find that $-\frac{1}{u} = \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} + c$.

You do need to give the final answer in terms of y, and use the initial condition. The final answer is:

$$y = \frac{3\left(1+x^2\right)^{\frac{1}{2}}}{4-\left(1+x^2\right)^{\frac{3}{2}}}\,.$$

(ii) This time use $y = u(1+x^3)^{\frac{1}{3}}$. The final answer is:

$$y = \frac{4\left(1+x^3\right)^{\frac{1}{3}}}{5-\left(1+x^3\right)^{\frac{4}{3}}}.$$

(iii) This part requires you to look at the cases $n=2$ (part **(i)**) and $n=3$ (part **(iii)**) and generalise in terms of n. Final answer:

$$y = \frac{(n+1)\left(1+x^n\right)^{\frac{1}{n}}}{(n+2)-\left(1+x^n\right)^{\frac{n+1}{n}}}.$$

Note: in each case equivalent forms would be acceptable, but it is easier to find the general case if you simplify the $n=2$ and $n=3$ cases as shown.

5 If you substitute $y = \mathrm{e}^x$ into equation $(*)$ you get:

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = (x-1)\mathrm{e}^x - x\mathrm{e}^x + \mathrm{e}^x = 0$$

and so $y = \mathrm{e}^x$ is a solution of $(*)$.

When substituting $y = u\mathrm{e}^x$, remember that u is a function of x so we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\mathrm{e}^x + \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{e}^x$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left(u\mathrm{e}^x + \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{e}^x\right) + \left(\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{e}^x + \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\mathrm{e}^x\right).$$

You can now substitute these into $(*)$, and you can divide throughout by e^x (this is OK as we have $\mathrm{e}^x \neq 0$). With some simplification you should obtain $(**)$.

Setting $\frac{\mathrm{d}u}{\mathrm{d}x} = v$ gives the equation $\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{x-2}{x-1}$. By using partial fractions[1] we get $\frac{x-2}{x-1} = 1 - \frac{1}{x-1}$. Integrating gives:

$$\ln|v| = -x + \ln|x-1| + c$$

and so:

$$v = k\mathrm{e}^{-x}(x-1) \qquad \text{where} \qquad k = \mathrm{e}^c\,.$$

We now have $\frac{\mathrm{d}u}{\mathrm{d}v} = k\mathrm{e}^{-x}(x-1)$ and so we have:

$$u = \int k\mathrm{e}^{-x}(x-1)\mathrm{d}x$$

and you can use integration by parts to obtain $u = -kx\mathrm{e}^{-x} + c'$.

The final instruction is a “Hence”, so you need to use the previous results. In particular setting $y = Ax + B\mathrm{e}^x$ and showing that this satisfies $(*)$ will not gain you any credit. Since $y = u\mathrm{e}^x$ we have $y = -kx + c'\mathrm{e}^x$ and so if we let $A = -k$ and $B = c'$ we know that $y = Ax + B\mathrm{e}^x$ satisfies $(*)$.

[1] Or we can note that $\frac{x-2}{x-1} = \frac{x-1-1}{x-1}$.