

STEP Support Programme

STEP 2 Calculus Questions: Solutions

These are not fully worked solutions — you need to fill in the gaps. It is a good idea to look at the “Hints” document before this one.

1 For the first part, $x = -1, 0, 1$.

We need $P'(x) - 2xP(x) \equiv kx(x-a)(x+a)(x-b)(x+b)$, though we actually could have $P'(x) - 2xP(x)$ having factors of $(x-a)^2$ etc., but we might as well use the simplest case! This means that $P(x)$ must be a quartic, so we can write $P(x) = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon$. You can then use $P'(x) - 2xP(x) \equiv x(x^2 - a^2)(x^2 - b^2)$ and equate coefficients to find the values of the coefficients of $P(x)$. Finding α , β and δ is fairly easy, γ and ϵ are a little harder.

$$P(x) = -\frac{x^4}{2} + \left(\frac{a^2}{2} + \frac{b^2}{2} - 1\right)x^2 + \left(\frac{a^2}{2} + \frac{b^2}{2} - \frac{a^2b^2}{2} - 1\right).$$

2 You should start by showing that

$$I = \int_0^\pi xf(\sin x)dx = \int_0^\pi (\pi - t)f(\sin t)dt.$$

You can then show that $2I = \int_0^\pi (x + \pi - x)f(\sin x)dx$.

You should expect to use this repeatedly in the rest of the question!

(i) If you use $u = \cos x$ then you should find that the integral is equal to $\frac{1}{2}\pi \int_{-1}^1 \frac{1}{4-u^2} du$.

Using partial fractions we have $\frac{1}{4-u^2} = \frac{A}{2+u} + \frac{B}{2-u}$ and then you can find A and B .

The final answer is $\frac{1}{4}\pi \ln 3$.

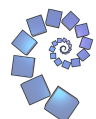
(ii) Split the integral into two parts, and then use a substitution of $t = x - \pi$, which will give you:

$$\int_0^\pi \frac{x \sin x}{3 + \sin^2 x} dx + \int_0^\pi \frac{(t + \pi) \sin(t + \pi)}{3 + \sin^2(t + \pi)} dt.$$

You can then use the fact that $\sin(t + \pi) = -\sin t$ to reduce the integrals to $-\pi \int_0^\pi \frac{\sin t}{3 + \sin^2 t}$ and, by using your work from part (i) the answer is $-\frac{1}{2}\pi \ln 3$.

(iii) Using $\sin 2x = 2 \sin x \cos x$, the stem result and a substitution $u = \cos x$ leads to the integral $\frac{1}{2}\pi \int_{-1}^1 \frac{2|u|}{4-u^2} du$ which is the same as $\pi \int_0^1 \frac{2u}{4-u^2} du$.

The result $\int \frac{f'(x)}{f(x)} = \ln |f(x)| + c$ may be useful. Final answer $\pi \ln \frac{4}{3}$.



- 3** (i) The first two answers should be $\frac{1}{\sqrt{3+x^2}}$ and $\frac{2x^2+3}{\sqrt{3+x^2}}$.

Then we can write

$$\sqrt{3+x^2} = \frac{1}{2} \left(\frac{3+2x^2}{\sqrt{3+x^2}} + \frac{3}{\sqrt{3+x^2}} \right)$$

and hence use the previous two answers to find $\int \sqrt{3+x^2} \, dx$. Don't forget the constant of integration!

Final answer: $\frac{1}{2}x\sqrt{3+x^2} + \frac{3}{2} \ln(x + \sqrt{3+x^2}) + c$.

- (ii) We can treat this as a quadratic equation, which leads to two differential equations:

$$\frac{dy}{dx} = \frac{-x \pm \sqrt{x^2+3}}{3}.$$

Integrating and using the given initial condition then gives:

$$y_1 = -\frac{1}{6}x^2 + \frac{1}{6}x\sqrt{3+x^2} + \frac{1}{2} \ln(x + \sqrt{3+x^2}) - \frac{1}{6} - \frac{1}{2} \ln 3$$

and

$$y_2 = -\frac{1}{6}x^2 - \frac{1}{6}x\sqrt{3+x^2} - \frac{1}{2} \ln(x + \sqrt{3+x^2}) + \frac{1}{2} + \frac{1}{2} \ln 3.$$



- 4 (i) With the given substitution we have $\frac{dy}{dx} = (1+x^2)^{\frac{1}{2}} \times \frac{du}{dx} + xu(1+x^2)^{-\frac{1}{2}}$.

This should reduce the differential equation to one where the variables are separable. The integral in x can be tackled with a substitution, or by inspection (“guessing” what the answer is and then checking it). You should find that $-\frac{1}{u} = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + c$.

You do need to give the final answer in terms of y , and use the initial condition. The final answer is:

$$y = \frac{3(1+x^2)^{\frac{1}{2}}}{4 - (1+x^2)^{\frac{3}{2}}}.$$

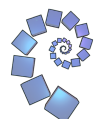
- (ii) This time use $y = u(1+x^3)^{\frac{1}{3}}$. The final answer is:

$$y = \frac{4(1+x^3)^{\frac{1}{3}}}{5 - (1+x^3)^{\frac{4}{3}}}.$$

- (iii) This part requires you to look at the cases $n = 2$ (part (i)) and $n = 3$ (part (ii)) and generalise in terms of n . Final answer:

$$y = \frac{(n+1)(1+x^n)^{\frac{1}{n}}}{(n+2) - (1+x^n)^{\frac{n+1}{n}}}.$$

Note: in each case equivalent forms would be acceptable, but it is easier to find the general case if you simplify the $n = 2$ and $n = 3$ cases as shown.



5 If you substitute $y = e^x$ into equation (*) you get:

$$(x - 1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = (x - 1)e^x - xe^x + e^x = 0$$

and so $y = e^x$ is a solution of (*).

When substituting $y = ue^x$, remember that u is a function of x so we have

$$\frac{dy}{dx} = ue^x + \frac{du}{dx}e^x$$

and

$$\frac{d^2y}{dx^2} = \left(ue^x + \frac{du}{dx}e^x\right) + \left(\frac{du}{dx}e^x + \frac{d^2u}{dx^2}e^x\right).$$

You can now substitute these into (*), and you can divide throughout by e^x (this is OK as we have $e^x \neq 0$). With some simplification you should obtain (**).

Setting $\frac{du}{dx} = v$ gives the equation $\frac{1}{v}\frac{dv}{dx} = -\frac{x-2}{x-1}$. By using partial fractions¹ we get $\frac{x-2}{x-1} = 1 - \frac{1}{x-1}$. Integrating gives:

$$\ln|v| = -x + \ln|x-1| + c$$

and so:

$$v = ke^{-x}(x-1) \quad \text{where} \quad k = e^c.$$

We now have $\frac{du}{dv} = ke^{-x}(x-1)$ and so we have:

$$u = \int ke^{-x}(x-1)dx$$

and you can use integration by parts to obtain $u = -kxe^{-x} + c'$.

The final instruction is a ‘‘Hence’’, so you need to use the previous results. In particular setting $y = Ax + Be^x$ and showing that this satisfies (*) will not gain you any credit. Since $y = ue^x$ we have $y = -kx + c'e^x$ and so if we let $A = -k$ and $B = c'$ we know that $y = Ax + Be^x$ satisfies (*).

¹Or we can note that $\frac{x-2}{x-1} = \frac{x-1-1}{x-1}$.

