STEP Support Programme

STEP 3 Complex Numbers: Hints

1 The 4 roots of $x^4 = 1$ are equispaced around a unit circle centre $(0, 0)$ with the first one at $(1, 0)$. If you evaluate a, a^2, a^3 for each you will see which ones are the primitive roots, and can then find $C_4(x) = (x - a_1)(x - a_2)$.

(i) For everything except $C_5(x)$ you can find the primitive roots and use these to find $C_i(x)$ without too much trouble. This technique will also work for $C_5(x)$ but there are 4 primitive roots in this case, and you will need to find $\cos\left(\frac{2\pi}{5}\right)$.

Another technique is to consider $x^n - 1 = 0$ and factorise out the *non*-primitive roots of this.

(ii) Trial and error will work here (and having already evaluated $C_1(x)$ to $C_6(x)$ helps with this). Another thing you can note is that the roots of $C_n(x)$ obey $x^4 = -1$ and hence we have $x^8 = 1$ so if $x^n = 1$ we have $n = 8k$ for some integer k.

(iii) Think about $C_2(x)$, $C_3(x)$ and $C_5(x)$. It might also be helpful to sketch the roots on an Argand diagram. If p is prime, how many primitive roots of $x^p = 1$ are there?

(iv) First show that a root of $C_n(x)$ cannot be a root of $C_m(x)$ if $m \neq n$.

2 A few facts that might help:

- $|p|^2 = pp^* = a^2$
- The vector PQ can be represented by the complex number $q - p$
- $\mathrm{i}z$ is a rotation of z by $\frac{1}{2}\pi$ anticlockwise
- Two things are perpendicular if one is a rotation of $\frac{1}{2}\pi$ and an enlargement of the other

When $n = 3$ set up three equations using $pq + rs = 0$ and then use them to find b_1 in terms of some other thing(s). Do the same when $n = 4$. You could try $n = 5$ before generalising.

3 There is quite a lot of work to be done in the "stem" of this question.

For the first result expand the brackets and use $e^{i\theta} = \cos\theta + i\sin\theta$.

The restriction $-\pi < \theta \leqslant \pi$ is useful in showing the required result for $z^{2n} + 1$. The roots will come in pairs of the form $\pm\theta$.

(i) This part is relatively short. If you make the substitution suggested and simplify things you should get the required result.

(ii) This part is a little trickier. If you substitute $z = i$, you will find that $1 + z^{2n} = 0$. This means that i is a root of $1 + z^{2n}$, and so $-i$ will be another root (as the polynomial has real coefficients). Hence there is a factor of $1 + z^2$ that you need to locate and factorise out.

4 The points will form an equilateral triangle if and only if you can rotate one point (β say) by 60° about a second point (which could be α) and land on the third point (γ). This shows that two lengths are the same (i.e. isosceles) and since the angle between these equal lengths is 60° and the other two are equal then they are all 60° and so the triangle is equilateral.

There are two conditions depending on whether the rotation is clockwise or anticlockwise. You can arrange these each in the form "*something*"$= 0$ and multiply them together.

If the roots of the equation are α, β and γ then we can write:

$$z^3 + az^2 + bz + c \equiv (z - \alpha)(z - \beta)(z - \gamma)\ .$$

Expand the brackets and you can find a, b (and c) in terms of α, β and γ.

For the last part the "low-tech" solution is to substitute $z = pw + q$ into $(*)$ and consider $A^2 - 3B$. There will be some messy algebra along the way!