1  2010 S3 Q3
For any given positive integer \( n \), a number \( a \) (which may be complex) is said to be a primitive \( n \)th root of unity if \( a^n = 1 \) and there is no integer \( m \) such that \( 0 < m < n \) and \( a^m = 1 \). Write down the two primitive 4th roots of unity.

Let \( C_n(x) \) be the polynomial such that the roots of the equation \( C_n(x) = 0 \) are the primitive \( n \)th roots of unity, the coefficient of the highest power of \( x \) is one and the equation has no repeated roots. Show that \( C_4(x) = x^2 + 1 \).

(i) Find \( C_1(x) \), \( C_2(x) \), \( C_3(x) \), \( C_5(x) \) and \( C_6(x) \), giving your answers as unfactorised polynomials.

(ii) Find the value of \( n \) for which \( C_n(x) = x^4 + 1 \).

(iii) Given that \( p \) is prime, find an expression for \( C_p(x) \), giving your answer as an unfactorised polynomial.

(iv) Prove that there are no positive integers \( q \), \( r \) and \( s \) such that \( C_q(x) \equiv C_r(x)C_s(x) \).

2  2007 S3 Q6
The distinct points \( P \), \( Q \), \( R \) and \( S \) in the Argand diagram lie on a circle of radius \( a \) centred at the origin and are represented by the complex numbers \( p \), \( q \), \( r \) and \( s \), respectively. Show that
\[
pq = -a^2 \frac{p - q}{p^* - q^*}.
\]
Deduce that, if the chords \( PQ \) and \( RS \) are perpendicular, then \( pq + rs = 0 \).

The distinct points \( A_1, A_2, \ldots, A_n \) (where \( n \geq 3 \)) lie on a circle. The points \( B_1, B_2, \ldots, B_n \) lie on the same circle and are chosen so that the chords \( B_1B_2, B_2B_3, \ldots, B_nB_1 \) are perpendicular, respectively, to the chords \( A_1A_2, A_2A_3, \ldots, A_nA_1 \). Show that, for \( n = 3 \), there are only two choices of \( B_1 \) for which this is possible. What is the corresponding result for \( n = 4 \)? State the corresponding results for values of \( n \) greater than 4.
3  2013 S3 Q4

Show that \((z - e^{i\theta})(z - e^{-i\theta}) = z^2 - 2z \cos \theta + 1\).

Write down the \((2n)\)th roots of \(-1\) in the form \(e^{i\theta}\), where \(-\pi < \theta \leq \pi\), and deduce that

\[
z^{2n} + 1 = \prod_{k=1}^{n} \left( z^2 - 2z \cos \left( \frac{(2k-1)\pi}{2n} \right) + 1 \right).
\]

Here, \(n\) is a positive integer, and the \(\prod\) notation denotes the product.

(i) By substituting \(z = i\) show that, when \(n\) is even,

\[
\cos \left( \frac{\pi}{2n} \right) \cos \left( \frac{3\pi}{2n} \right) \cos \left( \frac{5\pi}{2n} \right) \cdots \cos \left( \frac{(2n-1)\pi}{2n} \right) = (-1)^{\frac{1}{2}}n^{1-n}.
\]

(ii) Show that, when \(n\) is odd,

\[
\cos^2 \left( \frac{\pi}{2n} \right) \cos^2 \left( \frac{3\pi}{2n} \right) \cos^2 \left( \frac{5\pi}{2n} \right) \cdots \cos^2 \left( \frac{(n-2)\pi}{2n} \right) = n2^{1-n}.
\]

You may use without proof the fact that \(1 + z^{2n} = (1 + z^2)(1 - z^2 + z^4 - \cdots + z^{2n-2})\) when \(n\) is odd.

4  2006 S3 Q5

Show that the distinct complex numbers \(\alpha\), \(\beta\) and \(\gamma\) represent the vertices of an equilateral triangle (in clockwise or anti-clockwise order) if and only if

\[
\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta = 0.
\]

Show that the roots of the equation

\[
z^3 + az^2 + bz + c = 0 \quad (*)
\]

represent the vertices of an equilateral triangle if and only if \(a^2 = 3b\).

Under the transformation \(z = pw + q\), where \(p\) and \(q\) are given complex numbers with \(p \neq 0\), the equation \((*)\) becomes

\[
w^3 + Aw^2 + Bw + C = 0. \quad (**)\]

Show that if the roots of equation \((*)\) represent the vertices of an equilateral triangle, then the roots of equation \((**)\) also represent the vertices of an equilateral triangle.