STEP Support Programme

STEP 3 Complex Numbers Questions

1 **2010 S3 Q3**

For any given positive integer n, a number a (which may be complex) is said to be a *primitive nth root of unity* if $a^n = 1$ and there is no integer m such that $0 < m < n$ and $a^m = 1$. Write down the two primitive 4th roots of unity.

Let $\mathrm{C}_n(x)$ be the polynomial such that the roots of the equation $\mathrm{C}_n(x) = 0$ are the primitive nth roots of unity, the coefficient of the highest power of x is one and the equation has no repeated roots. Show that $\mathrm{C}_4(x) = x^2 + 1$.

(i) Find $\mathrm{C}_1(x)$, $\mathrm{C}_2(x)$, $\mathrm{C}_3(x)$, $\mathrm{C}_5(x)$ and $\mathrm{C}_6(x)$, giving your answers as unfactorised polynomials.

(ii) Find the value of n for which $\mathrm{C}_n(x) = x^4 + 1$.

(iii) Given that p is prime, find an expression for $\mathrm{C}_p(x)$, giving your answer as an unfactorised polynomial.

(iv) Prove that there are no positive integers q, r and s such that $\mathrm{C}_q(x) \equiv \mathrm{C}_r(x)\mathrm{C}_s(x)$.

2 **2007 S3 Q6**

The distinct points P, Q, R and S in the Argand diagram lie on a circle of radius a centred at the origin and are represented by the complex numbers p, q, r and s, respectively. Show that

$$pq = -a^2 \frac{p - q}{p^* - q^*}\,.$$

Deduce that, if the chords PQ and RS are perpendicular, then $pq + rs = 0$.

The distinct points A_1, A_2, $\ldots$, A_n (where $n \geqslant 3$) lie on a circle. The points B_1, B_2, $\ldots$, B_n lie on the same circle and are chosen so that the chords B_1B_2, B_2B_3, $\ldots$, B_nB_1 are perpendicular, respectively, to the chords A_1A_2, A_2A_3, $\ldots$, A_nA_1. Show that, for $n = 3$, there are only two choices of B_1 for which this is possible. What is the corresponding result for $n = 4$? State the corresponding results for values of n greater than 4.

3 2013 S3 Q4

Show that $(z - \mathrm{e}^{\mathrm{i}\theta})(z - \mathrm{e}^{-\mathrm{i}\theta}) = z^2 - 2z\cos\theta + 1$.

Write down the $(2n)$th roots of -1 in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leqslant \pi$, and deduce that

$$z^{2n} + 1 = \prod_{k=1}^{n} \left(z^2 - 2z\cos\left(\frac{(2k-1)\pi}{2n}\right) + 1 \right).$$

Here, n is a positive integer, and the $\prod$ notation denotes the product.

(i) By substituting $z = \mathrm{i}$ show that, when n is even,

$$\cos\left(\frac{\pi}{2n}\right)\cos\left(\frac{3\pi}{2n}\right)\cos\left(\frac{5\pi}{2n}\right)\cdots\cos\left(\frac{(2n-1)\pi}{2n}\right) = (-1)^{\frac{1}{2}n}2^{1-n}.$$

(ii) Show that, when n is odd,

$$\cos^2\left(\frac{\pi}{2n}\right)\cos^2\left(\frac{3\pi}{2n}\right)\cos^2\left(\frac{5\pi}{2n}\right)\cdots\cos^2\left(\frac{(n-2)\pi}{2n}\right) = n2^{1-n}.$$

You may use without proof the fact that $1 + z^{2n} = (1 + z^2)(1 - z^2 + z^4 - \cdots + z^{2n-2})$ when n is odd.

4 2006 S3 Q5

Show that the distinct complex numbers α, β and γ represent the vertices of an equilateral triangle (in clockwise or anti-clockwise order) if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta = 0.$$

Show that the roots of the equation

$$z^3 + az^2 + bz + c = 0 \qquad (*)$$

represent the vertices of an equilateral triangle if and only if $a^2 = 3b$.

Under the transformation $z = pw + q$, where p and q are given complex numbers with $p \neq 0$, the equation $(*)$ becomes

$$w^3 + Aw^2 + Bw + C = 0. \qquad (**)$$

Show that if the roots of equation $(*)$ represent the vertices of an equilateral triangle, then the roots of equation $(**)$ also represent the vertices of an equilateral triangle.