STEP Support Programme

STEP 3 Complex Numbers: Solutions

1 A *primitive* nth root of unity is one that will "generate" all the other roots. This means that a is a primitive root of unity iff[1] a, a^2, a^3, ..., $a^n = 1$ are all different and are the n roots of unity.[2]

The 4th roots of unity are $1, -1, \mathrm{i}, -\mathrm{i}$ and the primitive 4th roots of unity are i and $-$i; note that $\mathrm{i}^2 = -1, \mathrm{i}^3 = -\mathrm{i}, \mathrm{i}^4 = 1$ and $(-\mathrm{i})^2 = -1, (-\mathrm{i})^3 = \mathrm{i}, (-\mathrm{i})^4 = 1$ so both i and $-$i "generate" all the roots of $x^4 = 1$ unlike -1 and 1.

Hence $C_4(x) = (x - \mathrm{i})(x + \mathrm{i}) = x^2 - \mathrm{i}^2 = x^2 + 1$.

(i)
- There is only one root of $x^1 = 1$, and that is $x = 1$. Hence $C_1(x) = x - 1$.
- There are two roots of $x^2 = 1$, i.e. $x = \pm 1$. Hence $C_2(x) = x - (-1) = x + 1$.
- There are three roots of $x^3 = 1$, which are $1, \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$ and $\mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}}$. The last two are both primitive roots so:

$$\begin{aligned} C_3(x) &= \left(x - \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\right)\left(x - \mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}}\right) \\ &= x^2 - \left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} + \mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}}\right)x + 1 \\ &= x^2 - 2\cos\left(\tfrac{2\pi}{3}\right)x + 1 \\ &= x^2 + x + 1 \end{aligned}$$

  Alternatively you could have written the roots as $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$ and used:

$$\begin{aligned} C_3(x) &= \left(x + \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\mathrm{i}\right)\left(x + \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\mathrm{i}\right) \\ &= \left(x + \tfrac{1}{2}\right)^2 - \tfrac{3}{4}\mathrm{i}^2 \\ &= x^2 + x + \tfrac{1}{4} + \tfrac{3}{4} \end{aligned}$$

- There are five roots of $x^5 = 1$, which are $1, \mathrm{e}^{\pm\frac{2\pi \mathrm{i}}{5}}, \mathrm{e}^{\pm\frac{4\pi \mathrm{i}}{5}}$. This means that $C_5(x) = \left(x - \mathrm{e}^{\frac{2\pi \mathrm{i}}{5}}\right)\left(x - \mathrm{e}^{-\frac{2\pi \mathrm{i}}{5}}\right)\left(x - \mathrm{e}^{\frac{4\pi \mathrm{i}}{5}}\right)\left(x - \mathrm{e}^{-\frac{4\pi \mathrm{i}}{5}}\right)$. I don't fancy expanding this, and I will probably have to find $\cos\left(\frac{2\pi}{5}\right)$ which I don't know.[3] For now I will park this one.
- There are six roots of $x^6 = 1$. Drawing a sketch will show that the only primitive ones are $x = \mathrm{e}^{\frac{2\pi \mathrm{i}}{6}}$ and $x = \mathrm{e}^{-\frac{2\pi \mathrm{i}}{6}}$. Following the same method as for $C_3(x)$ gives $C_6(x) = x^2 - x + 1$, the only difference being that we use $\cos\left(\frac{2\pi}{6}\right) = \frac{1}{2}$ rather than $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$.

Coming back to $C_5(x)$. The 5th roots of unity all solve $x^5 - 1 = 0$. We can write this as $(x - 1)(x^4 + x^3 + x^2 + x + 1) = 0$. The only non-primitive root is $x = 1$ and this is accountable for the $x - 1$ factor. The other 4 (*primitive*) roots solve $x^4 + x^3 + x^2 + x + 1 = 0$ and so $C_5(x) = x^4 + x^3 + x^2 + x + 1$. The sort of method can be used for C_3 and C_6 (and others) as well.

[1] If and only if.

[2] If we have $a^p = a^q$ where $p \neq q$ and $0 < p < q \leqslant n$ then $a^{q-p} = 1$ and hence a is **not** a *primitive* nth root of unity.

[3] It can be calculated in various ways — try a web search.

(ii) We know that $n > 6$. There are 6 primitive roots of $x^7 = 1$, so $n \neq 7$. Consider $x^8 = 1$ — there are 8 roots of which 4 are primitive, and these are at $x = \mathrm{e}^{\pm\frac{\pi \mathrm{i}}{4}} = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}\mathrm{i}$ and $x = \mathrm{e}^{\pm\frac{3\pi \mathrm{i}}{4}} = -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}\mathrm{i}$. This gives:

$$\begin{aligned}
C_8(x) &= \left(x - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{i}\right)\left(x - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right)\left(x + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{i}\right)\left(x + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right) \\
&= \left(\left(x - \frac{1}{\sqrt{2}}\right)^2 - \frac{1}{2}\mathrm{i}^2\right)\left(\left(x + \frac{1}{\sqrt{2}}\right)^2 - \frac{1}{2}\mathrm{i}^2\right) \\
&= \left(x^2 - \frac{2}{\sqrt{2}}x + \frac{1}{2} + \frac{1}{2}\right)\left(x^2 + \frac{2}{\sqrt{2}}x + \frac{1}{2} + \frac{1}{2}\right) \\
&= \left(x^2 - \sqrt{2}x + 1\right)\left(x^2 + \sqrt{2}x + 1\right) \\
&= \left(x^2 + 1\right)^2 - 2x^2 \\
&= x^4 + 2x^2 + 1 - 2x^2 \\
&= x^4 + 1
\end{aligned}$$

and so $n = 8$.

A different approach would be to start by noting that $C_n(x) = 0 \implies x^4 = -1 \implies x^8 = 1$ and so n must be a multiple of 8. You would still have to verify that $C_8(x) = x^4 + 1$.

(iii) If p is prime, then the only non-primitive root of $x^p = 1$ is $x = 1$. You could then write $C_p(x)$ as the product of p different brackets involving $\mathrm{e}^{\frac{2k\pi}{p}\mathrm{i}}$, but as the question asks for an unfactorised polynomial this is probably not the way to go. Comparing to $C_5(x)$ we have:

$$\begin{aligned}
x^p - 1 &= 0 \\
(x-1)\left(x^{p-1} + x^{p-2} + x^{p-3} + \ldots + x + 1\right) &= 0
\end{aligned}$$

and so $C_p(x) = x^{p-1} + x^{p-2} + x^{p-3} + \ldots + x + 1$.

(iv) From part **(i)** the functions $C_k(x)$ have the following roots:

- $C_1(x)$: root $x = 1$
- $C_2(x)$: root $x = -1$
- $C_3(x)$: roots $x = \mathrm{e}^{\pm\frac{2\pi}{3}\mathrm{i}}$
- $C_4(x)$: roots $x = \pm\mathrm{i}$
- $C_5(x)$: roots $x = \mathrm{e}^{\pm\frac{2\pi}{5}\mathrm{i}}$ and $x = \mathrm{e}^{\pm\frac{4\pi}{5}\mathrm{i}}$
- $C_6(x)$: roots $x = \mathrm{e}^{\pm\frac{2\pi}{6}\mathrm{i}}$

and from this it appears that no root of $C_m(x)$ is also a root of $C_n(x)$ for any $m \neq n$.

WLOG[4] let $m < n$. By the definition of $C_n(x)$, if a is a root of $C_n(x)$ then there can be no integer m (where $0 < m < n$) such that $a^m = 1$ and so if a is a root of $C_n(x)$ then a **cannot** be a root of $C_m(x)$.

[4] Without Loss of Generality.

Thus if we have $C_q(x) \equiv C_r(x)C_s(x)$ and $C_q(a) = 0$ (i.e. $x = a$ is a root of $C_q(x) = 0$) then we must have either $C_r(a) = 0$ or $C_s(a) = 0$. This means that either a is a root of $C_r(x)$ or $C_s(x)$ which means that we must have $q = r$ and $C_s(x) \equiv 1$ or $q = s$ and $C_r(x) \equiv 1$. This is not possible for positive integers q, r and s as there will always be a least one root of $C_k(x)$ if k is a positive integer, hence $C_k(x) \not\equiv 1$. Hence there are no positive integers q, r and s such that $C_q(x) \equiv C_r(x)C_s(x)$.

2 Since P and Q lie on a circle radius a centre the origin we know that $|p| = |q| = a$ and so $pp^* = qq^* = a^2$. We then have:

$$\begin{aligned} a^2(p-q) &= a^2p - a^2q \\ &= qq^*p - pp^*q \\ &= pq(q^* - p^*) \\ &= -pq(p^* - q^*) \end{aligned}$$

Hence $pq = -a^2 \dfrac{p-q}{p^*-q^*}$ as required.

You can also do the first part of the question by setting $p = a(\cos\alpha + \mathrm{i}\sin\alpha)$ and $q = a(\cos\beta + \mathrm{i}\sin\beta)$, and then considering pq and $\dfrac{p-q}{p^*-q^*}$. This will take a bit more effort.

If PQ and RQ are perpendicular then we can write $p - q = \mathrm{i}k(r-s)$ for some real k. Note that multiplying by i has the affect of rotating anticlockwise by $\frac{1}{2}\pi$. If k is negative then PQ will be a rotation of $\frac{1}{2}\pi$ clockwise, and the magnitude of k will give the ratio of the lengths PQ and RS. Hence we have:

$$p - q = \mathrm{i}k(r-s) \qquad \text{and} \qquad p^* - q^* = -\mathrm{i}k(r^* - s^*)$$

and so:

$$\begin{aligned} pq &= -a^2 \frac{p-q}{p^*-q^*} \\ &= -a^2 \frac{\mathrm{i}k(r-s)}{-\mathrm{i}k(r^*-s^*)} \\ &= a^2 \frac{r-s}{r^*-s^*} \\ &= -rs \end{aligned}$$

Therefore $pq = -rs \implies pq + rs = 0$.

Note that the points $A_1, A_2, \cdots, A_n$ are **fixed**, where as $B_1, B_2, \cdots, B_n$ are chosen.

When $n = 3$ we have $B_1B_2 \perp A_1A_2$, $B_2B_3 \perp A_2A_3$ and $B_3B_1 \perp A_3A_1$. Therefore we have $a_1a_2 + b_1b_2 = 0$ etc. This gives:

$$\begin{aligned} b_1b_2 \times b_3b_1 &= a_1a_2 \times a_3a_1 \\ b_1^2 &= \frac{a_1a_2 \times a_3a_1}{b_2b_3} \\ b_1^2 &= -\frac{a_1a_2 \times a_3a_1}{a_2a_3} \\ b_1^2 &= -a_1^2 \end{aligned}$$

and so $b_1 = \pm \mathrm{i}a_1$.

Note that none of the numbers a_j, b_k can be equal to 0 as the points all lie on a circle radius a (which we can assume to be greater than 0) and the circle is centred on the origin.

When $n = 4$ we have:

$$\begin{aligned} b_1b_2 &= -a_1a_2 \\ b_2b_3 &= -a_2a_3 \\ b_3b_4 &= -a_3a_4 \\ b_4b_1 &= -a_4a_1 \end{aligned}$$

Multiplying these 4 together gives:

$$b_1^2 = \frac{a_1^2 a_2^2 a_3^2 a_4^2}{b_2^2 b_3^2 b_4^2}$$

At this point we can either use $b_2 b_3 = -a_2 a_3$ or $b_3 b_4 = -a_3 a_4$ to cancel some terms, but in either case b_1 is related to another b value and so the choice of b_1 is not restricted despite the A points being fixed. For example, if we use $b_2 b_3 = -a_2 a_3$ we have

$$b_1^2 = \frac{a_1^2 a_4^2}{b_4^2}$$

and so the value of b_1 will depend on what value of b_4 is picked.

When n is odd, let $n = 2m+1$ and we have:

$$\begin{aligned} b_1^2 &= -a_1^2 \times \frac{a_2^2 a_3^2}{b_2^2 b_3^2} \times \frac{a_4^2 a_5^4}{b_4^2 b_5^2} \times \cdots \times \frac{a_{2m}^2 a_{2m+1}^2}{b_{2m}^2 b_{2m+1}^2} \\ &= -a_1^2 \\ \implies b_1 &= \pm \mathrm{i} a_1 \end{aligned}$$

and so b_1 is restricted to 2 choices.

When n is even (let $n = 2m$), we have:

$$\begin{aligned} b_1^2 &= -a_1^2 \times \frac{a_2^2 a_3^2}{b_2^2 b_3^2} \times \frac{a_4^2 a_5^4}{b_4^2 b_5^2} \times \cdots \times \frac{a_{2m}^2}{b_{2m}^2} \\ &= -a_1^2 \times \frac{a_{2m}^2}{b_{2m}^2} \end{aligned}$$

and so b_1 can be arbitrarily picked.

3 We have:

$$\begin{aligned}(z-\mathrm{e}^{\mathrm{i}\theta})(z-\mathrm{e}^{-\mathrm{i}\theta}) &= z^2 - z\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}\right)+\mathrm{e}^{(\mathrm{i}\theta-\mathrm{i}\theta)}\\ &= z^2 - z\left(\cos\theta+\mathrm{i}\sin\theta+\cos\theta-\mathrm{i}\sin\theta\right)+1\\ &= z^2-2z\cos\theta+1\end{aligned}$$

-1 can be written as $\mathrm{e}^{\mathrm{i}\pi}$. This means that the $(2n)$th roots of -1 are:

$$\mathrm{e}^{\mathrm{i}\left(\frac{\pi+2m\pi}{2n}\right)} = \mathrm{e}^{\mathrm{i}\left(\frac{2m+1}{2n}\right)\pi}$$

We could take $0 \leqslant m \leqslant 2n-1$, but as the question says it wants the roots in a form with $-\pi < \theta \leqslant \pi$ it would be better to take m in the range $-n \leqslant m \leqslant n-1$. This gives the roots in the form $\mathrm{e}^{\mathrm{i}\theta}$ where $\theta = \left(\frac{-2n+1}{2n}\right)\pi, \left(\frac{-2n+3}{2n}\right)\pi, \cdots, \left(\frac{2n-3}{2n}\right)\pi, \left(\frac{2n-1}{2n}\right)\pi$. Noting that these come in pairs we can write the roots as $\mathrm{e}^{\pm\mathrm{i}\left(\frac{2m+1}{2n}\right)\pi}$ where $0 \leqslant m \leqslant n-1$.

It is sometimes easier to find the roots by drawing a diagram. You should be able to see that one root makes an angle of $\frac{\pi}{2n}$ with the positive real axis and then there are $2n$ of these equally spaced around the origin.

Using this we have:

$$\begin{aligned}z^{2n}+1 &= \prod_{m=0}^{n-1}\left(z-\mathrm{e}^{\mathrm{i}\left(\frac{2m+1}{2n}\right)\pi}\right)\left(z-\mathrm{e}^{-\mathrm{i}\left(\frac{2m+1}{2n}\right)\pi}\right)\\ &= \prod_{k=1}^{n}\left(z-\mathrm{e}^{\mathrm{i}\left(\frac{2k-1}{2n}\right)\pi}\right)\left(z-\mathrm{e}^{-\mathrm{i}\left(\frac{2k-1}{2n}\right)\pi}\right) \quad \text{where } k=m+1\\ &= \prod_{k=1}^{n}\left(z^2-2z\cos\left(\frac{(2k-1)\pi}{2n}\right)+1\right) \quad \text{using the first result}\end{aligned}$$

(i) When $z=\mathrm{i}$ we have $z^{2n}=(z^2)^n=(-1)^n$, so if n is even, $z^{2n}=1$. Using the result shown in the stem we have:

$$\begin{aligned}z^{2n}+1 &= \prod_{k=1}^{n}\left(z^2-2z\cos\left(\frac{(2k-1)\pi}{2n}\right)+1\right)\\ 2 &= \prod_{k=1}^{n}\left(\cancel{(-1)}-2\mathrm{i}\cos\left(\frac{(2k-1)\pi}{2n}\right)\cancel{+1}\right)\\ 2 &= (-1)^n2^n\mathrm{i}^n\prod_{k=1}^{n}\cos\left(\frac{(2k-1)\pi}{2n}\right)\\ 2 &= 1\times 2^n\times(-1)^{\frac{1}{2}n}\times\cos\left(\frac{\pi}{2n}\right)\times\cos\left(\frac{3\pi}{2n}\right)\times\cos\left(\frac{5\pi}{2n}\right)\times\cdots\times\cos\left(\frac{(2n-1)\pi}{2n}\right)\end{aligned}$$

Therefore:

$$\begin{aligned}\cos\left(\tfrac{\pi}{2n}\right)\cos\left(\tfrac{3\pi}{2n}\right)\cos\left(\tfrac{5\pi}{2n}\right)\cdots\cos\left(\frac{(2n-1)\pi}{2n}\right) &= \frac{2}{2^n\times(-1)^{\frac{1}{2}n}}\\ &= 2^{1-n}(-1)^{\frac{1}{2}n}\end{aligned}$$

(ii) At first glance this question looks very similar to the last one. However, if n is odd then $z^{2n}+1=(-1)^n+1=0$, which is slightly problematic.

Considering the RHS of the second result of the stem we have:

$$\prod_{k=1}^{n}\left(z^2-2z\cos\left(\tfrac{(2k-1)\pi}{2n}\right)+1\right)=\left(z^2-2z\cos\left(\tfrac{\pi}{2n}\right)+1\right)\times$$
$$\left(z^2-2z\cos\left(\tfrac{3\pi}{2n}\right)+1\right)\times\cdots\times\left(z^2-2z\cos\left(\tfrac{(n-2)\pi}{2n}\right)+1\right)$$
$$\times\left(z^2-2z\cos\left(\tfrac{n\pi}{2n}\right)+1\right)\times\left(z^2-2z\cos\left(\tfrac{(n+2)\pi}{2n}\right)+1\right)$$
$$\times\cdots\times\left(z^2-2z\cos\left(\tfrac{(2n-1)\pi}{2n}\right)+1\right)$$

The arguments are in the form $\frac{\text{odd number}\times\pi}{2n}$, so if n is odd one will be $\frac{n\pi}{2n}$.

Since $\cos\left(\frac{n\pi}{2n}\right)=\cos\frac{1}{2}\pi=0$, we have $z^2-2z\cos\left(\frac{n\pi}{2n}\right)+1=z^2+1$. Using $z^{2n}+1=(z^2+1)(1-z^2+z^4-\cdots+z^{2n-2})$ gives us:

$$(1-z^2+z^4-\cdots+z^{2n-2})=\left(z^2-2z\cos\left(\tfrac{\pi}{2n}\right)+1\right)\times$$
$$\left(z^2-2z\cos\left(\tfrac{3\pi}{2n}\right)+1\right)\times\cdots\times\left(z^2-2z\cos\left(\tfrac{(n-2)\pi}{2n}\right)+1\right)\times$$
$$\left(z^2-2z\cos\left(\tfrac{(n+2)\pi}{2n}\right)+1\right)\times\cdots\times\left(z^2-2z\cos\left(\tfrac{(2n-1)\pi}{2n}\right)+1\right)$$

Then substituting $z=\mathrm{i}$ gives:

$$(1-\mathrm{i}^2+\mathrm{i}^4+\cdots+(-1)^{n-1})=\left((-1)-2\mathrm{i}\cos\left(\tfrac{\pi}{2n}\right)+1\right)\times$$
$$\left((-1)-2\mathrm{i}\cos\left(\tfrac{3\pi}{2n}\right)+1\right)\times\cdots\times\left((-1)-2\mathrm{i}\cos\left(\tfrac{(n-2)\pi}{2n}\right)+1\right)$$
$$\times\left((-1)-2\mathrm{i}\cos\left(\tfrac{(n+2)\pi}{2n}\right)+1\right)\times\cdots\times$$
$$\left((-1)-2\mathrm{i}\cos\left(\tfrac{(2n-1)\pi}{2n}\right)+1\right)$$
$$n=(-2\mathrm{i})^{n-1}\times\cos\left(\frac{\pi}{2n}\right)\times\cos\left(\frac{3\pi}{2n}\right)\times\cdots$$
$$\times\cos\left(\frac{(n-2)\pi}{2n}\right)\times\cos\left(\frac{(n+2)\pi}{2n}\right)\times\cdots\times\cos\left(\frac{(2n-1)\pi}{2n}\right)$$

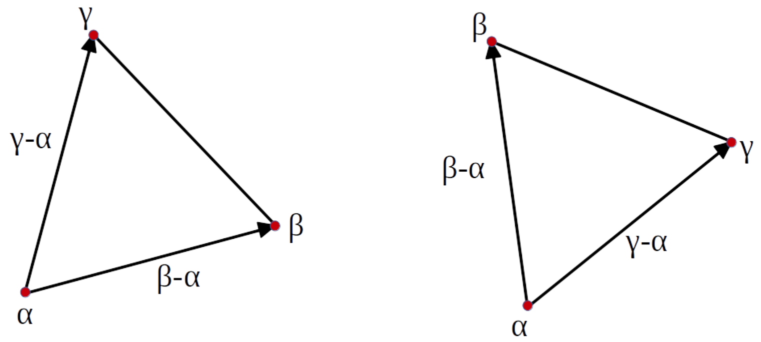
Noting that $\mathrm{i}^{n-1}=(-1)^{\frac{n-1}{2}}$ and that $\cos(\pi-\alpha)=-\cos\alpha$ we have:

$$n=(-1)^{n-1}2^{n-1}(-1)^{\frac{n-1}{2}}\times\cos\left(\frac{\pi}{2n}\right)\cos\left(\frac{3\pi}{2n}\right)\cdots\cos\left(\frac{(n-2)\pi}{2n}\right)$$
$$\times(-1)^{\frac{n-1}{2}}\cos\left(\frac{\pi}{2n}\right)\cos\left(\frac{3\pi}{2n}\right)\cdots\cos\left(\frac{(n-2)\pi}{2n}\right)$$
$$=(-1)^{n-1}2^{n-1}(-1)^{\frac{n-1}{2}}(-1)^{\frac{n-1}{2}}\times\cos^2\left(\frac{\pi}{2n}\right)\cos^2\left(\frac{3\pi}{2n}\right)\cdots\cos^2\left(\frac{(n-2)\pi}{2n}\right)$$

Then, using $(-1)^{n-1}=1$ we have:

$$n2^{1-n}=\cos^2\left(\frac{\pi}{2n}\right)\cos^2\left(\frac{3\pi}{2n}\right)\cdots\cos^2\left(\frac{(n-2)\pi}{2n}\right)\quad\text{as required.}$$

4 α, β and γ are the three vertices of a triangle then then we have one of the cases below:



Then the triangle is equilateral if and only if the point γ is a rotation of point β about point α of $\frac{\pi}{3}$ radians either clockwise or anti-clockwise.[5]

This gives:

$$\begin{aligned}\gamma - \alpha &= \mathrm{e}^{\frac{i\pi}{3}}\left(\beta - \alpha\right) \qquad \text{or} \\ \gamma - \alpha &= \mathrm{e}^{-\frac{i\pi}{3}}\left(\beta - \alpha\right)\end{aligned} \tag{*}$$

This is true if and only if:

$$\begin{aligned}\left[\gamma - \alpha - \mathrm{e}^{\frac{i\pi}{3}}\left(\beta-\alpha\right)\right]\left[\gamma - \alpha - \mathrm{e}^{-\frac{i\pi}{3}}\left(\beta-\alpha\right)\right] &= 0\\ \left(\gamma-\alpha\right)^2 - \left(\mathrm{e}^{\frac{i\pi}{3}} + \mathrm{e}^{-\frac{i\pi}{3}}\right)\left(\beta-\alpha\right)\left(\gamma-\alpha\right) + \left(\beta-\alpha\right)^2 &= 0\\ \left(\gamma-\alpha\right)^2 - \left(2\cos\left(\tfrac{\pi}{3}\right)\right)\left(\beta-\alpha\right)\left(\gamma-\alpha\right) + \left(\beta-\alpha\right)^2 &= 0\\ \left(\gamma^2 - 2\gamma\alpha + \alpha^2\right) - \left(\beta-\alpha\right)\left(\gamma-\alpha\right) + \left(\beta^2 - 2\beta\alpha + \alpha^2\right) &= 0\\ \gamma^2 - 2\gamma\alpha + \alpha^2 - \left(\beta\gamma - \beta\alpha - \alpha\gamma + \alpha^2\right) + \beta^2 - 2\beta\alpha + \alpha^2 &= 0\\ \alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta &= 0\end{aligned}$$

The third line of the working above uses $\cos\theta = \frac{1}{2}\left(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}\right)$.

To convince myself that $(*)$ is true (geometry not being my strongest area) I had to do a little work. I know that multiplying by $\mathrm{e}^{i\theta}$ represents a rotation of θ anti-clockwise *about the origin*.

To justify $(*)$ I translated the triangle so that the point represented by α was at the origin, so that the other two points are now at $\beta - \alpha$ and $\gamma - \alpha$. I can now deduce the result $\gamma - \alpha = \mathrm{e}^{\pm\frac{i\pi}{3}}\left(\beta - \alpha\right)$ as this is now a case of translating about the origin.

Alternatively I could have used that a rotation of point z by angle θ about point c is given $\mathrm{e}^{i\theta}(z - c) + c$. This gives the point γ as $\gamma = \mathrm{e}^{\pm\frac{i\pi}{3}}\left(\beta - \alpha\right) + \alpha$, which is equivalent to $(*)$.

[5]This then means that the triangle is isosceles as lengths $|\beta - \alpha|$ and $|\gamma - \alpha|$ are equal and since the angle between these equal sides is 60° then the other two angles are also 60° and hence the triangle is equilateral.

Let the roots of the equation be α, β and γ. Then $(z-\alpha)(z-\beta)(z-\gamma) \equiv z^3 + az^2 + bz + c$ gives us:

$$\begin{aligned} a &= -\left(\alpha+\beta+\gamma\right) \\ b &= \alpha\beta+\beta\gamma+\gamma\alpha \\ c &= -\alpha\beta\gamma \end{aligned}$$

Then we have:

$$\begin{aligned} a^2 - 3b &= \left(\alpha+\beta+\gamma\right)^2 - 3\left(\alpha\beta+\beta\gamma+\gamma\alpha\right) \\ &= \alpha^2+\beta^2+\gamma^2+2\beta\gamma+2\gamma\alpha+2\alpha\beta - 3\left(\alpha\beta+\beta\gamma+\gamma\alpha\right) \\ &= \alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta \end{aligned}$$

Hence $a^2 = 3b$ if and only if $\alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta = 0$, so the roots form an equilateral triangle if and only if $a^2 = 3b$.

Substituting $z = pw + q$ into $z^3 + az^2 + bz + c = 0$ gives:

$$\begin{aligned} (pw+q)^3 + a(pw+q)^2 + b(pw+q) + c &= 0 \\ p^3w^3 + 3p^2qw^2 + 3pq^2w + q^3 + a\left(p^2w^2+2pqw+q^2\right) + b(pw+q) + c &= 0 \\ p^3w^3 + \left(3p^2q + ap^2\right)w^2 + \left(3pq^2+2apq+bp\right)w + \left(q^3+aq^2+bq+c\right) &= 0 \\ w^3 + \frac{\left(3p^2q+ap^2\right)}{p^3}w^2 + \frac{\left(3pq^2+2apq+bp\right)}{p^3}w + \frac{\left(q^3+aq^2+bq+c\right)}{p^3} &= 0 \end{aligned}$$

Note that $p \neq 0$ so we can divide by p^3.

Then we have:

$$\begin{aligned} A^2 - 3B &= \frac{1}{p^6}\left(9p^4q^2 + 6ap^4q + a^2p^4\right) - \frac{3}{p^3}\left(3pq^2 + 2apq + bp\right) \\ &= \frac{1}{p^3}\left(9pq^2 + 6apq + a^2p - 9pq^2 - 6apq - 3bp\right) \\ &= \frac{1}{p^3}\left(a^2p - 3bp\right) \\ &= \frac{1}{p^2}\left(a^2 - 3b\right) \end{aligned}$$

So $a^2 - 3b = 0 \implies A^2 - 3B = 0$, and so if the roots of $z^3 + az^2 + bz + c = 0$ represent the vertices of an equilateral triangle then the roots of $w^3 + Aw^2 + Bw + C = 0$ also represent the vertices of an equilateral triangle.

Alternatively, we could argue that the transformation $w \mapsto pw$ is a rotation and an enlargement, so the triangle formed after this transformation is similar to the original one. Then the transformation $pw \mapsto pw + q$ is a translation, so the triangle here is congruent to the one before.

Hence the triangles which has vertices w_1, w_2, w_3 and those with vertices z_1, z_2, z_3 where $z_i = pw_i + q$ are similar, and hence if the triangle with vertices z_1, z_2, z_3 is equilateral then so is the one with vertices w_1, w_2, w_3. This argument does have the advantage that there is less algebra to go wrong with!

In the solutions provided by the Admissions Testing Service and by MEI it is stated without proof that the triangle is equilateral if and only if:

$$\beta - \gamma = \omega(\gamma - \alpha) \qquad \text{or}$$
$$\beta - \gamma = \omega^2(\gamma - \alpha) \qquad (*)$$

where ω is the cube root of unity equal to $\dfrac{-1+\sqrt{3}}{2}$.

We then have that the triangle is equilateral if and only if:

$$[\beta - \gamma - \omega(\gamma - \alpha)] \times \left[\beta - \gamma - \omega^2(\gamma - \alpha)\right] = 0$$
$$[\beta - \gamma(1+\omega) + \alpha\omega] \times \left[\beta - \gamma\left(1+\omega^2\right) + \alpha\omega^2\right] = 0$$
$$\left[\beta + \gamma\omega^2 + \alpha\omega\right] \times \left[\beta + \gamma\omega + \alpha\omega^2\right] = 0$$

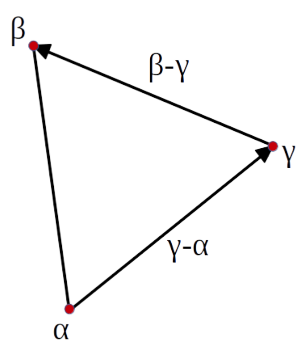
This last step uses the fact that $1+\omega+\omega^2 = 0$. This can be seen by considering the cube roots of unity geometrically or by noting that the cube roots satisfy $\omega^3 - 1 = 0$, which can be written as $(\omega - 1)\left(\omega^2+\omega+1\right) = 0$.

Expanding the brackets gives us:

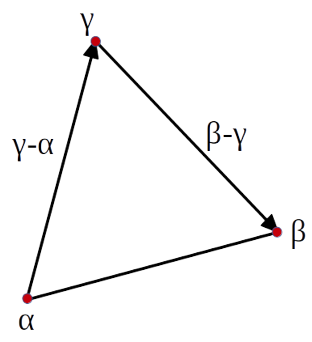
$$\beta^2 + \gamma^2\omega^3 + \alpha^2\omega^3 + \beta\gamma\left(\omega+\omega^2\right) + \alpha\beta\left(\omega+\omega^2\right) + \gamma\alpha\left(\omega+\omega^2\right) = 0$$
$$\beta^2 + \gamma^2 + \alpha^2 - \beta\gamma - \alpha\beta - \gamma\alpha = 0$$

Where the last step uses $\omega^3 = 1$ and $1+\omega+\omega^2 = 0$.

The fact $(*)$ was stated just as a fact, and it appears that no justification is necessary. However, I wanted to convince myself why this is true before using it. The diagram below shows the two different cases.



β-γ = ω(γ-α)



β-γ = ω²(γ-α)

For the first one, the triangle is equilateral if and only if point β is formed by translating point γ by 120° anticlockwise about α and then translating it by $\gamma - \alpha$. Using $z \mapsto e^{i\theta}(z-c)+c$ for a rotation of θ anti clockwise about point c and that $e^{i\frac{2\pi}{3}} = \omega$ gives us:

$$\beta = \omega(\gamma - \alpha) + \alpha + (\gamma - \alpha) \qquad \text{i.e.}$$
$$\beta - \gamma = \omega(\gamma - \alpha)$$

A similar argument can be used for the other triangle, but this time the rotation is $120°$ **clockwise** or equivalently $240°$ anti-clockwise, which is represented by multiplication by ω^2.

Having done this I then worked through this part of the question again, using the working shown above. I decided that I preferred my method, so presented that as the given "solution" here!

STEP III 2008 Q7 has another complex number question involving equilateral triangles.