STEP Support Programme

STEP 3 Complex Numbers Topic Notes

Euler's formula:

$$e^{i\theta} \equiv \cos\theta + i\sin\theta$$

This means that we can write $z = r\cos\theta + ir\sin\theta$ as $z = re^{i\theta}$ (known as **exponential form**).

From this result we can deduce that:

$$\cos\theta = \tfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad \text{and} \qquad \sin\theta = \tfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Summation of Trigonometric series

For example: to find $\cos\theta + \cos 2\theta + \ldots + \cos n\theta = \displaystyle\sum_{k=1}^{n} \cos(k\theta)$ consider:

$$\begin{aligned}
\sum_{k=1}^{n}\Big[\cos(k\theta) + i\sin(k\theta)\Big] &= \sum_{k=1}^{n} e^{ik\theta} \\
&= \frac{e^{i\theta}\left(e^{in\theta} - 1\right)}{e^{i\theta} - 1} && \textit{Geometric series sum} \\
&= e^{i\theta}\frac{e^{\frac{1}{2}in\theta}\left(e^{\frac{1}{2}in\theta} - e^{-\frac{1}{2}in\theta}\right)}{e^{\frac{1}{2}i\theta}\left(e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}\right)} \\
&= e^{\frac{1}{2}i(n+1)\theta}\frac{\cancel{2i}\sin\left(\frac{1}{2}n\theta\right)}{\cancel{2i}\sin\left(\frac{1}{2}\theta\right)}
\end{aligned}$$

Then equating real parts gives us:

$$\sum_{k=1}^{n}\cos(k\theta) = \cos\left(\frac{1}{2}(n+1)\theta\right) \times \frac{\sin\left(\frac{1}{2}n\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}$$

Multiplication and division. Often easiest in exponential form!

If we have $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ then:

$$\begin{aligned}
z_1z_2 &= r_1r_2e^{i(\theta_1+\theta_2)} \\
\frac{z_1}{z_2} &= \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}
\end{aligned}$$

I.e. when multiplying two numbers the moduli are multiplied and the arguments added together and when dividing one complex number by another, the modulus of the first is divided by the modulus of the second and the argument of the second is subtracted from the argument of the first.

de Moivre's theorem

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

where n is an integer. This can be proved by induction on n (such as is shown here) or can be deduced from **Euler's formula** using $\left(\mathrm{e}^{\mathrm{i}\theta}\right)^n = \mathrm{e}^{\mathrm{i}n\theta}$.

This can be used to deduce trigonometrical identities, e.g.:

$$\begin{aligned}\cos 5\theta + \mathrm{i}\sin 5\theta &= (\cos\theta + \mathrm{i}\sin\theta)^5 \\ &= \mathrm{c}^5 + 5\mathrm{i}\mathrm{c}^4\mathrm{s} - 10\mathrm{c}^3\mathrm{s}^2 - 10\mathrm{i}\mathrm{c}^2\mathrm{s}^3 + 5\mathrm{c}\mathrm{s}^4 + \mathrm{i}\mathrm{s}^5\end{aligned}$$

Where $\mathrm{c} = \cos\theta$ and $\mathrm{s} = \sin\theta$. Equating imaginary parts gives us:

$$\begin{aligned}\sin 5\theta &= 5\mathrm{c}^4\mathrm{s} - 10c^2s^3 + \mathrm{s}^5 \\ &= 5\left(1 - \mathrm{s}^2\right)^2\mathrm{s} - 10\left(1 - s^2\right)\mathrm{s}^3 + \mathrm{s}^5 \\ &= 16\mathrm{s}^5 - 20\mathrm{s}^3 + 5s\end{aligned}$$

nth roots

If $z = r\mathrm{e}^{\mathrm{i}\theta}$ then the nth roots of n are given by:

$$r^{1/n}\mathrm{e}^{\mathrm{i}\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)} \quad \text{where} \quad 0 \leqslant k \leqslant n-1$$

If we like we could change the range of k, e.g. $-4 \leqslant k \leqslant n-5$.

These n roots will form the vertices of a regular n-sided polygon in the Argand plane.

In particular, the nth roots of unity are given by:

$$\mathrm{e}^{\frac{2\mathrm{i}k\pi}{n}}$$

It can be helpful to draw a sketch of where the nth roots must be in the Argand plane, and then use this to deduce what they are.