

## STEP Support Programme

### STEP 2 Complex Numbers Questions

#### 1 1992 S2 Q10

Let  $\alpha$  be a fixed angle,  $0 < \alpha \leq \frac{1}{2}\pi$ . In each of the following cases, sketch the locus of  $z$  in the Argand diagram (the complex plane):

(i)  $\arg\left(\frac{z-1}{z}\right) = \alpha,$

(ii)  $\arg\left(\frac{z-1}{z}\right) = \alpha - \pi,$

(iii)  $\left|\frac{z-1}{z}\right| = 1.$

Let  $z_1, z_2, z_3$  and  $z_4$  be four points lying (in that order) on a circle in the Argand diagram. If

$$w = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_4 - z_1)(z_2 - z_3)}$$

show, by considering  $\arg w$ , that  $w$  is real.

#### 2 2000 S2 Q4

Prove that

$$(\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi) = \cos(\theta + \phi) + i \sin(\theta + \phi)$$

and that, for every positive integer  $n$ ,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

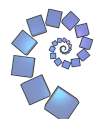
By considering  $(5 - i)^2(1 + i)$ , or otherwise, prove that

$$\arctan\left(\frac{7}{17}\right) + 2 \arctan\left(\frac{1}{5}\right) = \frac{\pi}{4}.$$

Prove also that

$$3 \arctan\left(\frac{1}{4}\right) + \arctan\left(\frac{1}{20}\right) + \arctan\left(\frac{1}{1985}\right) = \frac{\pi}{4}.$$

[Note that  $\arctan \theta$  is another notation for  $\tan^{-1} \theta$ .]



**3 2011 S3 Q8**

The complex numbers  $z$  and  $w$  are related by

$$w = \frac{1 + iz}{i + z}.$$

Let  $z = x + iy$  and  $w = u + iv$ , where  $x, y, u$  and  $v$  are real. Express  $u$  and  $v$  in terms of  $x$  and  $y$ .

- (i) By setting  $x = \tan(\theta/2)$ , or otherwise, show that if the locus of  $z$  is the real axis  $y = 0$ ,  $-\infty < x < \infty$ , then the locus of  $w$  is the circle  $u^2 + v^2 = 1$  with one point omitted.
- (ii) Find the locus of  $w$  when the locus of  $z$  is the line segment  $y = 0$ ,  $-1 < x < 1$ .
- (iii) Find the locus of  $w$  when the locus of  $z$  is the line segment  $x = 0$ ,  $-1 < y < 1$ .
- (iv) Find the locus of  $w$  when the locus of  $z$  is the line  $y = 1$ ,  $-\infty < x < \infty$ .

**4 2005 S3 Q8**

In this question,  $a$  and  $c$  are distinct non-zero complex numbers. The complex conjugate of any complex number  $z$  is denoted by  $z^*$ .

Show that

$$|a - c|^2 = aa^* + cc^* - ac^* - ca^*$$

and hence prove that the triangle  $OAC$  in the Argand diagram, whose vertices are represented by  $0, a$  and  $c$  respectively, is right angled at  $A$  if and only if  $2aa^* = ac^* + ca^*$ .

Points  $P$  and  $P'$  in the Argand diagram are represented by the complex numbers  $ab$  and  $\frac{a}{b^*}$ , where  $b$  is a non-zero complex number. A circle in the Argand diagram has centre  $C$  and passes through the point  $A$ , and is such that  $OA$  is a tangent to the circle. Show that the point  $P$  lies on the circle if and only if the point  $P'$  lies on the circle.

Conversely, show that if the points represented by the complex numbers  $ab$  and  $\frac{a}{b^*}$ , for some non-zero complex number  $b$  with  $bb^* \neq 1$ , both lie on a circle centre  $C$  in the Argand diagram which passes through  $A$ , then  $OA$  is a tangent to the circle.

