STEP Support Programme

STEP 2 Complex Numbers Questions

1 1992 S2 Q10

Let α be a fixed angle, $0 < \alpha \leqslant \frac{1}{2}\pi$. In each of the following cases, sketch the locus of z in the Argand diagram (the complex plane):

(i) $\arg\left(\dfrac{z-1}{z}\right) = \alpha,$

(ii) $\arg\left(\dfrac{z-1}{z}\right) = \alpha - \pi,$

(iii) $\left|\dfrac{z-1}{z}\right| = 1.$

Let z_1, z_2, z_3 and z_4 be four points lying (in that order) on a circle in the Argand diagram. If

$$w = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_4 - z_1)(z_2 - z_3)}$$

show, by considering $\arg w$, that w is real.

2 2000 S2 Q4

Prove that

$$(\cos\theta + \mathrm{i}\sin\theta)(\cos\phi + \mathrm{i}\sin\phi) = \cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi)$$

and that, for every positive integer n,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta.$$

By considering $(5-\mathrm{i})^2(1+\mathrm{i})$, or otherwise, prove that

$$\arctan\left(\frac{7}{17}\right) + 2\arctan\left(\frac{1}{5}\right) = \frac{\pi}{4}\,.$$

Prove also that

$$3\arctan\left(\frac{1}{4}\right) + \arctan\left(\frac{1}{20}\right) + \arctan\left(\frac{1}{1985}\right) = \frac{\pi}{4}\,.$$

[Note that $\arctan\theta$ is another notation for $\tan^{-1}\theta$.]

3 **2011 S3 Q8**

The complex numbers z and w are related by

$$w = \frac{1 + \mathrm{i}z}{\mathrm{i} + z}\,.$$

Let $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$, where x, y, u and v are real. Express u and v in terms of x and y.

(i) By setting $x = \tan(\theta/2)$, or otherwise, show that if the locus of z is the real axis $y = 0$, $-\infty < x < \infty$, then the locus of w is the circle $u^2 + v^2 = 1$ with one point omitted.

(ii) Find the locus of w when the locus of z is the line segment $y = 0$, $-1 < x < 1$.

(iii) Find the locus of w when the locus of z is the line segment $x = 0$, $-1 < y < 1$.

(iv) Find the locus of w when the locus of z is the line $y = 1$, $-\infty < x < \infty$.

4 **2005 S3 Q8**

In this question, a and c are distinct non-zero complex numbers. The complex conjugate of any complex number z is denoted by z^*.

Show that

$$|a - c|^2 = aa^* + cc^* - ac^* - ca^*$$

and hence prove that the triangle OAC in the Argand diagram, whose vertices are represented by 0, a and c respectively, is right angled at A if and only if $2aa^* = ac^* + ca^*$.

Points P and P' in the Argand diagram are represented by the complex numbers ab and $\dfrac{a}{b^*}$, where b is a non-zero complex number. A circle in the Argand diagram has centre C and passes through the point A, and is such that OA is a tangent to the circle. Show that the point P lies on the circle if and only if the point P' lies on the circle.

Conversely, show that if the points represented by the complex numbers ab and $\dfrac{a}{b^*}$, for some non-zero complex number b with $bb^* \neq 1$, both lie on a circle centre C in the Argand diagram which passes through A, then OA is a tangent to the circle.