STEP Support Programme

STEP III Complex Numbers: Solutions

1 A *primitive* nth root of unity is one that will "generate" all the other roots. This means that a is a primitive root of unity iff[1] a, a^2, a^3, ..., $a^n = 1$ are all different and are the n roots of unity.[2]

The 4th roots of unity are $1, -1, i, -i$ and the primitive 4th roots of unity are i and $-i$; note that $i^2 = -1, i^3 = -i, i^4 = 1$ and $(-i)^2 = -1, (-i)^3 = i, (-i)^4 = 1$ so both i and $-i$ "generate" all the roots of $x^4 = 1$ unlike -1 and 1.

Hence $C_4(x) = (x - i)(x + i) = x^2 - i^2 = x^2 + 1$.

(i)

- There is only one root of $x^1 = 1$, and that is $x = 1$. Hence $C_1(x) = x - 1$.
- There are two roots of $x^2 = 1$, i.e. $x = \pm 1$. Hence $C_2(x) = x - (-1) = x + 1$.
- There are three roots of $x^3 = 1$, which are $1, \mathrm{e}^{\frac{2\pi i}{3}}$ and $\mathrm{e}^{-\frac{2\pi i}{3}}$. The last two are both primitive roots so:

$$\begin{aligned}
C_3(x) &= \left(x - \mathrm{e}^{\frac{2\pi i}{3}}\right)\left(x - \mathrm{e}^{-\frac{2\pi i}{3}}\right) \\
&= x^2 - \left(\mathrm{e}^{\frac{2\pi i}{3}} + \mathrm{e}^{-\frac{2\pi i}{3}}\right)x + 1 \\
&= x^2 - 2\cos\left(\tfrac{2\pi}{3}\right)x + 1 \\
&= x^2 + x + 1
\end{aligned}$$

  Alternatively you could have written the roots as $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ and used:

$$\begin{aligned}
C_3(x) &= \left(x + \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)\left(x + \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right) \\
&= \left(x + \tfrac{1}{2}\right)^2 - \tfrac{3}{4}i^2 \\
&= x^2 + x + \tfrac{1}{4} + \tfrac{3}{4}
\end{aligned}$$

- There are five roots of $x^5 = 1$, which are $1, \mathrm{e}^{\pm\frac{2\pi i}{5}}, \mathrm{e}^{\pm\frac{4\pi i}{5}}$. This means that $C_5(x) = \left(x - \mathrm{e}^{\frac{2\pi i}{5}}\right)\left(x - \mathrm{e}^{-\frac{2\pi i}{5}}\right)\left(x - \mathrm{e}^{\frac{4\pi i}{5}}\right)\left(x - \mathrm{e}^{-\frac{4\pi i}{5}}\right)$. I don't fancy expanding this, and I will probably have to find $\cos\left(\frac{2\pi}{5}\right)$ which I don't know.[3] For now I will park this one.
- There are six roots of $x^6 = 1$. Drawing a sketch will show that the only primitive ones are $x = \mathrm{e}^{\frac{2\pi i}{6}}$ and $x = \mathrm{e}^{-\frac{2\pi i}{6}}$. Following the same method as for $C_3(x)$ gives $C_6(x) = x^2 - x + 1$, the only difference being that we use $\cos\left(\frac{2\pi}{6}\right) = \frac{1}{2}$ rather than $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$.

Coming back to $C_5(x)$. The 5th roots of unity all solve $x^5 - 1 = 0$. We can write this as $(x - 1)(x^4 + x^3 + x^2 + x + 1) = 0$. The only non-primitive root is $x = 1$ and this is accountable for the $x - 1$ factor. The other 4 (*primitive*) roots solve $x^4 + x^3 + x^2 + x + 1 = 0$ and so $C_5(x) = x^4 + x^3 + x^2 + x + 1$. The sort of method can be used for C_3 and C_6 (and others) as well.

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[1] If and only if.

[2] If we have $a^p = a^q$ where $p \neq q$ and $0 < p < q \leqslant n$ then $a^{q-p} = 1$ and hence a is **not** a *primitive* nth root of unity.

[3] It can be calculated in various ways — try a web search.

(ii) We know that $n > 6$. There are 6 primitive roots of $x^7 = 1$, so $n \neq 7$. Consider $x^8 = 1$ — there are 8 roots of which 4 are primitive, and these are at $x = \mathrm{e}^{\pm\frac{\pi i}{4}} = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$ and $x = \mathrm{e}^{\pm\frac{3\pi i}{4}} = -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$. This gives:

$$\begin{aligned}
C_8(x) &= \left(x - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)\left(x - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)\left(x + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)\left(x + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)\\
&= \left(\left(x - \frac{1}{\sqrt{2}}\right)^2 - \frac{1}{2}i^2\right)\left(\left(x + \frac{1}{\sqrt{2}}\right)^2 - \frac{1}{2}i^2\right)\\
&= \left(x^2 - \frac{2}{\sqrt{2}}x + \frac{1}{2} + \frac{1}{2}\right)\left(x^2 + \frac{2}{\sqrt{2}}x + \frac{1}{2} + \frac{1}{2}\right)\\
&= \left(x^2 - \sqrt{2}x + 1\right)\left(x^2 + \sqrt{2}x + 1\right)\\
&= \left(x^2 + 1\right)^2 - 2x^2\\
&= x^4 + 2x^2 + 1 - 2x^2\\
&= x^4 + 1
\end{aligned}$$

and so $n = 8$.

A different approach would be to start by noting that $C_n(x) = 0 \implies x^4 = -1 \implies x^8 = 1$ and so n must be a multiple of 8. You would still have to verify that $C_8(x) = x^4 + 1$.

(iii) If p is prime, then the only non-primitive root of $x^p = 1$ is $x = 1$. You could then write $C_p(x)$ as the product of p different brackets involving $\mathrm{e}^{\frac{2k\pi}{p}i}$, but as the question asks for an unfactorised polynomial this is probably not the way to go. Comparing to $C_5(x)$ we have:

$$\begin{aligned}
x^p - 1 &= 0\\
(x-1)\left(x^p + x^{p-1} + x^{p-2} + \ldots + x + 1\right) &= 0
\end{aligned}$$

and so $C_p(x) = x^p + x^{p-1} + x^{p-2} + \ldots + x + 1$.

(iv) From part **(i)** the functions $C_i(x)$ have the following roots:

- $C_1(x)$: root $x = 1$
- $C_2(x)$: root $x = -1$
- $C_3(x)$: roots $x = \mathrm{e}^{\pm\frac{2\pi}{3}i}$
- $C_4(x)$: roots $x = \pm i$
- $C_5(x)$: roots $x = \mathrm{e}^{\pm\frac{2\pi}{5}i}$ and $x = \mathrm{e}^{\pm\frac{4\pi}{5}i}$
- $C_6(x)$: roots $x = \mathrm{e}^{\pm\frac{2\pi}{6}i}$

and from this it appears that no root of $C_m(x)$ is also a root of $C_n(x)$ for any $m \neq n$.

WLOG[4] let $m < n$. By the definition of $C_n(x)$, if a is a root of $C_n(x)$ then there can be no integer m (where $0 < m < n$) such that $a^m = 1$ and so if a is a root of $C_n(x)$ then a **cannot** be a root of $C_m(x)$.

[4] Without Loss of Generality.

Thus if we have $C_q(x) \equiv C_r(x)C_s(x)$ and $C_q(a) = 0$ (i.e. $x = a$ is a root of $C_q(x) = 0$) then we must have either $C_r(a) = 0$ or $C_s(a) = 0$. This means that either a is a root of $C_r(x)$ or $C_s(x)$ which means that we must have $q = r$ and $C_s(x) \equiv 1$ or $q = s$ and $C_r(x) \equiv 1$. This is not possible for positive integers q, r and s as there will always be a least one root of $C_i(x)$ if i is a positive integer, hence $C_i(x) \not\equiv 1$. Hence there are no positive integers q, r and s such that $C_q(x) \equiv C_r(x)C_s(x)$.

2 Substituting into w gives us:

$$\begin{aligned}
w &= \frac{1+i(x+iy)}{i+(x+iy)} \\
&= \frac{1-y+ix}{x+i(1+y)} \\
&= \frac{\big(1-y+ix\big)\big(x-i(1+y)\big)}{\big(x+i(1+y)\big)\big(x-i(1+y)\big)} \\
&= \frac{x(1-\cancel{y})+x(1+\cancel{y})+i\left(x^2-(1-y)(1+y)\right)}{x^2+(1+y)^2} \\
&= \frac{2x+i\left(x^2+y^2-1\right)}{x^2+(1+y)^2}
\end{aligned}$$

Therefore we have $u=\dfrac{2x}{x^2+(1+y)^2}$ and $v=\dfrac{x^2+y^2-1}{x^2+(1+y)^2}$.

(i) Setting $x=\tan(\theta/2)$ and $y=0$ gives

$$\begin{aligned}
u &= \frac{2\tan\left(\frac{\theta}{2}\right)}{\tan^2\left(\frac{\theta}{2}\right)+1} \\
&= \frac{2\tan\left(\frac{\theta}{2}\right)\times\cos^2\left(\frac{\theta}{2}\right)}{\left(\tan^2\left(\frac{\theta}{2}\right)+1\right)\times\cos^2\left(\frac{\theta}{2}\right)} \\
&= \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)+\cos^2\left(\frac{\theta}{2}\right)} \\
&= \frac{\sin\theta}{1} \\
&= \sin\theta
\end{aligned}$$

and

$$\begin{aligned}
v &= \frac{\tan^2\left(\frac{\theta}{2}\right)-1}{\tan^2\left(\frac{\theta}{2}\right)+1} \\
&= \frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)\times\cos^2\left(\frac{\theta}{2}\right)}{\left(\tan^2\left(\frac{\theta}{2}\right)+1\right)\times\cos^2\left(\frac{\theta}{2}\right)} \\
&= \frac{\sin^2\left(\frac{\theta}{2}\right)-\cos^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)+\cos^2\left(\frac{\theta}{2}\right)} \\
&= -\cos\theta
\end{aligned}$$

This means that we have $u^2+v^2=\sin^2\theta+\cos^2\theta=1$. However, $\tan\phi$ is undefined for $\phi=\frac{\pi}{2}$, so we need to exclude $\theta=\pi$, i.e. the point $(\sin\pi,-\cos\pi)=(0,1)$ is excluded.

(ii) This case is almost identical to the previous one except that x is restricted to $-1<x<1$. This means we need to restrict θ so that $-1<\tan\left(\frac{\theta}{2}\right)<1$ i.e. $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

Since $u=\sin\theta$ we have $-1<u<1$ and since $v=-\cos\theta$ we have $-1<v<0$. Hence the locus is the semi-circle which is part of $u^2+v^2=1$ which lies below the v-axis.

(iii) If we let $x = 0$ we get $u = 0$ and $v = \dfrac{y^2-1}{(1+y)^2} = \dfrac{y-1}{y+1}$. Hence we are restricted to the v-axis and as $-1 < y < 1$ we have $-\infty < v < 0$. Therefore the locus is the negative v-axis.

(iv) As this looks similar to part **(i)** we can try using $x = \tan\left(\frac{\theta}{2}\right)$ again. This gives:

$$u = \frac{2\tan\left(\frac{\theta}{2}\right)}{\tan^2\left(\frac{\theta}{2}\right)+4}$$

This doesn't look promising as the denominator will not simplify nicely. Instead try using $x = 2\tan\left(\frac{\theta}{2}\right)$, which gives:

$$\begin{aligned} u &= \frac{4\tan\left(\frac{\theta}{2}\right)}{4\tan^2\left(\frac{\theta}{2}\right)+4} \\ &= \frac{\tan\left(\frac{\theta}{2}\right)}{\tan^2\left(\frac{\theta}{2}\right)+1} \\ &= \frac{\left(\tan\left(\frac{\theta}{2}\right)\right)\times\cos^2\left(\frac{\theta}{2}\right)}{\left(\tan^2\left(\frac{\theta}{2}\right)+1\right)\times\cos^2\left(\frac{\theta}{2}\right)} \\ &= \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\ &= \tfrac{1}{2}\sin\theta \end{aligned}$$

We also get:

$$\begin{aligned} v &= \frac{4\tan^2\left(\frac{\theta}{2}\right)}{4\tan^2\left(\frac{\theta}{2}\right)+4} \\ &= \frac{\left(\tan^2\left(\frac{\theta}{2}\right)\right)\times\cos^2\left(\frac{\theta}{2}\right)}{\left(\tan^2\left(\frac{\theta}{2}\right)+1\right)\times\cos^2\left(\frac{\theta}{2}\right)} \\ &= \sin^2\left(\frac{\theta}{2}\right) \\ &= \tfrac{1}{2}\left(1-\cos\theta\right) \end{aligned}$$

where the last step uses $\cos 2A = 1 - 2\sin^2 A$.

Like in part **(i)** we cannot have $\theta = \pi$, so the point $\left(\frac{1}{2}\sin\pi, \frac{1}{2}\left(1-\cos\pi\right)\right) = (0,1)$ is excluded.

The final stage is to describe this locus fully. As we have $\sin\theta$ and $\cos\theta$ involved it is likely to be something circle related. We have:

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ (2u)^2 + (1-2v)^2 &= 1 \\ 4u^2 + 4\left(\tfrac{1}{2}-v\right)^2 &= 1 \\ u^2 + \left(\tfrac{1}{2}-v\right)^2 &= \tfrac{1}{4} \\ u^2 + \left(v-\tfrac{1}{2}\right)^2 &= \tfrac{1}{4} \end{aligned}$$

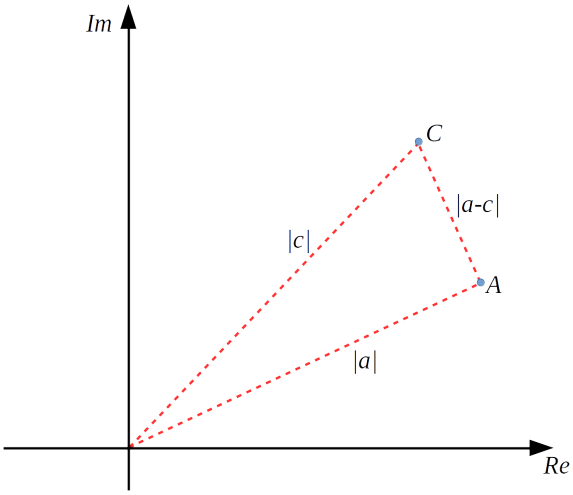
hence the locus is a circle radius $\frac{1}{2}$, centre $\left(0, \frac{1}{2}\right)$ with the point $(0,1)$ excluded.

I haven't included any sketches in this solution, but a very good way of making your answers clear would be to sketch the loci (make sure the labels on your axes are correct — u and v). You can use an empty circle to show the excluded point(s).

3 Using $|z|^2 = zz^*$ gives:

$$\begin{aligned}|a-c|^2 &= (a-c)(a^*-c^*)\\ &= aa^* + cc^* - ca^* - ac^*\end{aligned}$$

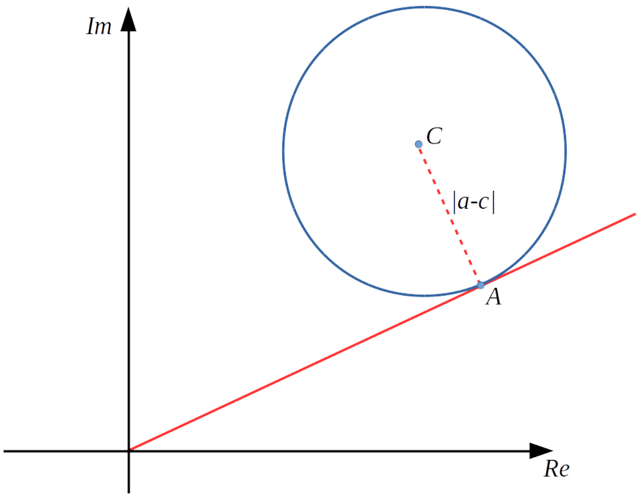
It is helpful to draw a diagram to show the points in the argand plane representing a and c:



Then the triangle has a right angle at A if and only if $|c|^2 = |a|^2 + |a-c|^2$, i.e.:

$$\begin{aligned}\iff \quad \cancel{cc^*} &= aa^* + aa^* + \cancel{cc^*} - ca^* - ac^*\\ \iff \quad 0 &= 2aa^* - ca^* - ac^*\\ \iff \quad 2aa^* &= ca^* + ac^*\end{aligned}$$

The circle and tangent could look something like:



The equation of the circle will be $|z-c|^2 = |a-c|^2$. This can be written as:

$$\begin{aligned}(z-c)(z^*-c^*) &= (a-c)(a^*-c^*)\\ zz^* + \cancel{cc^*} - cz^* - zc^* &= aa^* + \cancel{cc^*} - ca^* - ac^*\end{aligned}$$

Since the tangent is at right angles to the circle we have $2aa^* = ca^* + ac^*$ or $2aa^* - ca^* - ac^* = 0$ and so we can write the equation of the circle as $zz^* - zc^* - cz^* + aa^* = 0$.

P will lie on this circle if and only if:

$$\begin{aligned} (ab)(ab)^* - (ab)c^* - c(ab)^* + aa^* &= 0 \qquad \text{i.e.} \\ aba^*b^* - abc^* - ca^*b^* + aa^* &= 0 \qquad (*) \end{aligned}$$

And P' will lie on this circle if and only if:

$$\begin{aligned} \left(\frac{a}{b^*}\right)\left(\frac{a}{b^*}\right)^* - \left(\frac{a}{b^*}\right)c^* - c\left(\frac{a}{b^*}\right)^* + aa^* &= 0 \qquad \text{i.e.} \\ \frac{aa^*}{bb^*} - \frac{ac^*}{b^*} - \frac{ca^*}{b} + aa^* &= 0 \qquad \text{multiply by } bb^*(\neq 0) \text{ to give} \\ aa^* - abc^* - ca^*b^* + aa^*bb^* &= 0 \end{aligned}$$

This last line is the exactly same condition as $(*)$, so point P is on the circle if and only if point P' is on the circle.

For the converse we cannot assume that OA is a tangent, so we cannot use the condition $2aa^* = ac^* + ca^*$. Going back to $|z - c|^2 = |a - c|^2$ gives:

$$\begin{aligned} (z - c)(z^* - c^*) &= (z - a)(z^* - a^*) \\ zz^* + \cancel{cc^*} - cz^* - zc^* &= aa^* + \cancel{cc^*} - ca^* - ac^* \end{aligned}$$

If both P and P' lie on this circle then we have:

$$(ab)(ab)^* - c(ab)^* - (ab)c^* = aa^* - ca^* - ac^* \qquad \text{and} \qquad (1)$$

$$\left(\frac{a}{b^*}\right)\left(\frac{a}{b^*}\right)^* - c\left(\frac{a}{b^*}\right)^* - \left(\frac{a}{b^*}\right)c^* = aa^* - ca^* - ac^* \qquad \text{i.e.} \qquad (2)$$

$$aa^* - ca^*b^* - abc^* = (aa^* - ca^* - ac^*)\, bb^* \qquad (3)$$

If we can show that $2aa^* - ca^* - ac^* = 0$ then we will have shown that $\angle OAC = 90^\circ$ and hence that OA is perpendicular to AC and so the line OA is a tangent to the circle.

If we take equation (1) and add aa^* to both sides, and take equation (3) and add aa^*bb^* we get the equations:

$$aa^*bb^* - ca^*b^* - abc^* + aa^* = 2aa^* - ca^* - ac^* \qquad \text{and} \qquad (4)$$

$$aa^* - ca^*b^* - abc^* + aa^*bb^* = (2aa^* - ca^* - ac^*)\, bb^* \qquad (5)$$

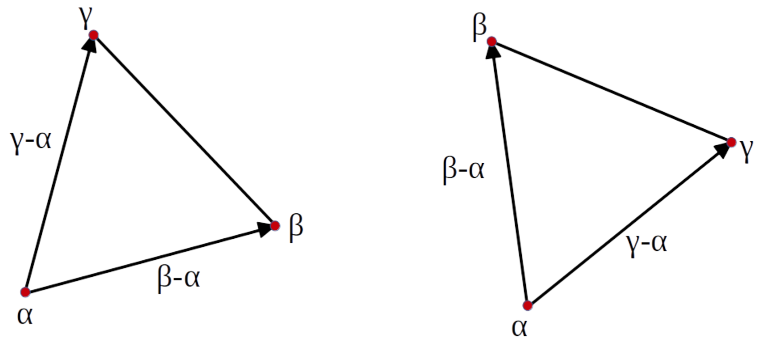
The left hand side of equations (4) and (5) are the same, so if we consider (5)-(4) we get:

$$\begin{aligned} 0 &= (2aa^* - ca^* - ac^*)\, bb^* - (2aa^* - ca^* - ac^*) \\ 0 &= (2aa^* - ca^* - ac^*)\, (bb^* - 1) \end{aligned}$$

and since we are told that $bb^* \neq 1$ we have $2aa^* - ca^* - ac^* = 0$ and the line OA is a tangent to the circle.

Note that if $bb^* = 1$ then we can write $b = \mathrm{e}^{i\theta}$. This means that $\dfrac{a}{b*} = \dfrac{a}{\mathrm{e}^{-i\theta}} = a\mathrm{e}^{i\theta} = ab$ and hence $P = P'$.

4 α, β and γ are the three vertices of a triangle then then we have one of the cases below:



Then the triangle is equilateral if and only if the point γ is a rotation of point β about point α of $\frac{\pi}{3}$ radians either clockwise or anti-clockwise.[5]

This gives:

$$\gamma - \alpha = \mathrm{e}^{\frac{i\pi}{3}}\left(\beta - \alpha\right) \qquad \text{or} \qquad (*)$$
$$\gamma - \alpha = \mathrm{e}^{-\frac{i\pi}{3}}\left(\beta - \alpha\right)$$

This is true if and only if:

$$\begin{aligned}
\left[\gamma - \alpha - \mathrm{e}^{\frac{i\pi}{3}}\left(\beta - \alpha\right)\right]\left[\gamma - \alpha - \mathrm{e}^{-\frac{i\pi}{3}}\left(\beta - \alpha\right)\right] &= 0\\
\left(\gamma - \alpha\right)^2 - \left(\mathrm{e}^{\frac{i\pi}{3}} + \mathrm{e}^{-\frac{i\pi}{3}}\right)\left(\beta - \alpha\right)\left(\gamma - \alpha\right) + \left(\beta - \alpha\right)^2 &= 0\\
\left(\gamma - \alpha\right)^2 - \left(2\cos\left(\tfrac{\pi}{3}\right)\right)\left(\beta - \alpha\right)\left(\gamma - \alpha\right) + \left(\beta - \alpha\right)^2 &= 0\\
\left(\gamma^2 - 2\gamma\alpha + \alpha^2\right) - \left(\beta - \alpha\right)\left(\gamma - \alpha\right) + \left(\beta^2 - 2\beta\alpha + \alpha^2\right) &= 0\\
\gamma^2 - 2\gamma\alpha + \alpha^2 - \left(\beta\gamma - \beta\alpha - \alpha\gamma + \alpha^2\right) + \beta^2 - 2\beta\alpha + \alpha^2 &= 0\\
\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta &= 0
\end{aligned}$$

The third line of the working above uses $\cos\theta = \frac{1}{2}\left(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}\right)$.

To convince myself that $(*)$ is true (geometry not being my strongest area) I had to do a little work. I know that multiplying by $\mathrm{e}^{i\theta}$ represents a rotation of θ anti-clockwise *about the origin*.

To justify $(*)$ I translated the triangle so that the point represented by α was at the origin, so that the other two points are now at $\beta - \alpha$ and $\gamma - \alpha$. I can now deduce the result $\gamma - \alpha = \mathrm{e}^{\pm\frac{i\pi}{3}}\left(\beta - \alpha\right)$ as this is now a case of translating about the origin.

Alternatively I could have used that a rotation of point z by angle θ about point c is given $\mathrm{e}^{i\theta}(z - c) + c$. This gives the point γ as $\gamma = \mathrm{e}^{\pm\frac{i\pi}{3}}\left(\beta - \alpha\right) + \alpha$, which is equivalent to $(*)$.

[5] This then means that the triangle is isosceles as lengths $|\beta - \alpha|$ and $|\gamma - \alpha|$ are equal and since the angle between these equal sides is 60° then the other two angles are also 60° and hence the triangle is equilateral.

Let the roots of the equation be α, β and γ. Then $(z-\alpha)(z-\beta)(z-\gamma) \equiv z^3 + az^2 + bz + c$ gives us:

$$\begin{aligned} a &= -\left(\alpha + \beta + \gamma\right) \\ b &= \alpha\beta + \beta\gamma + \gamma\alpha \\ c &= -\alpha\beta\gamma \end{aligned}$$

Then we have:

$$\begin{aligned} a^2 - 3b &= \left(\alpha + \beta + \gamma\right)^2 - 3\left(\alpha\beta + \beta\gamma + \gamma\alpha\right) \\ &= \alpha^2 + \beta^2 + \gamma^2 + 2\beta\gamma + 2\gamma\alpha + 2\alpha\beta - 3\left(\alpha\beta + \beta\gamma + \gamma\alpha\right) \\ &= \alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta \end{aligned}$$

Hence $a^2 = 3b$ if and only if $\alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta = 0$, so the roots form an equilateral triangle if and only if $a^2 = 3b$.

Substituting $z = pw + q$ into $z^3 + az^2 + bz + c = 0$ gives:

$$\begin{aligned} (pw+q)^3 + a(pw+q)^2 + b(pw+q) + c &= 0 \\ p^3w^3 + 3p^2qw^2 + 3pq^2w + q^3 + a\left(p^2w^2 + 2pqw + q^2\right) + b(pw+q) + c &= 0 \\ p^3w^3 + \left(3p^2q + ap^2\right)w^2 + \left(3pq^2 + 2apq + bp\right)w + \left(q^3 + aq^2 + bq + c\right) &= 0 \\ w^3 + \frac{\left(3p^2q + ap^2\right)}{p^3}w^2 + \frac{\left(3pq^2 + 2apq + bp\right)}{p^3}w + \frac{\left(q^3 + aq^2 + bq + c\right)}{p^3} &= 0 \end{aligned}$$

Note that $p \neq 0$ so we can divide by p^3.

Then we have:

$$\begin{aligned} A^2 - 3B &= \frac{1}{p^6}\left(9p^4q^2 + 6ap^4q + a^2p^4\right) - \frac{3}{p^3}\left(3pq^2 + 2apq + bp\right) \\ &= \frac{1}{p^3}\left(9pq^2 + 6apq + a^2p - 9pq^2 - 6apq - 3bp\right) \\ &= \frac{1}{p^3}\left(a^2p - 3bp\right) \\ &= \frac{1}{p^2}\left(a^2 - 3b\right) \end{aligned}$$

So $a^2 - 3b = 0 \implies A^2 - 3B = 0$, and so if the roots of $z^3 + az^2 + bz + c = 0$ represent the vertices of an equilateral triangle then the roots of $w^3 + Aw^2 + Bw + C = 0$ also represent the vertices of an equilateral triangle.

Alternatively, we could argue that the transformation $w \mapsto pw$ is a rotation and an enlargement, so the triangle formed after this transformation is similar to the original one. Then the transformation $pw \mapsto pw + q$ is a translation, so the triangle here is congruent to the one before.

Hence the triangles which has vertices w_1, w_2, w_3 and those with vertices z_1, z_2, z_3 where $z_i = pw_i + q$ are similar, and hence if the triangle with vertices z_1, z_2, z_3 is equilateral then so is the one with vertices w_1, w_2, w_3. This argument does have the advantage that there is less algebra to go wrong with!

In the solutions provided by the Admissions Testing Service and by MEI it is stated without proof that the triangle is equilateral if and only if:

$$\beta - \gamma = \omega\,(\gamma - \alpha) \qquad \text{or}$$
$$\beta - \gamma = \omega^2\,(\gamma - \alpha) \qquad (*)$$

where ω is the cube root of unity equal to $\dfrac{-1+\sqrt{3}}{2}$.

We then have that the triangle is equilateral if and only if:

$$\begin{aligned}
[\beta - \gamma - \omega\,(\gamma - \alpha)] \times \left[\beta - \gamma - \omega^2\,(\gamma - \alpha)\right] &= 0\\
[\beta - \gamma\,(1+\omega) + \alpha\omega] \times \left[\beta - \gamma\left(1+\omega^2\right) + \alpha\omega^2\right] &= 0\\
\left[\beta + \gamma\omega^2 + \alpha\omega\right] \times \left[\beta + \gamma\omega + \alpha\omega^2\right] &= 0
\end{aligned}$$

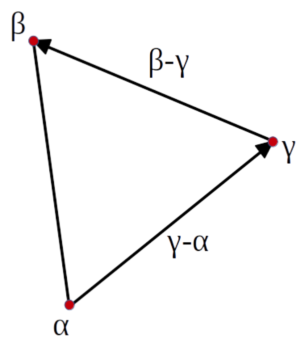
This last step uses the fact that $1 + \omega + \omega^2 = 0$. This can be seen by considering the cube roots of unity geometrically or by noting that the cube roots satisfy $\omega^3 - 1 = 0$, which can be written as $(\omega - 1)\left(\omega^2 + \omega + 1\right) = 0$.

Expanding the brackets gives us:

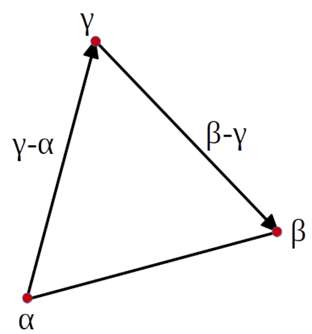
$$\begin{aligned}
\beta^2 + \gamma^2\omega^3 + \alpha^2\omega^3 + \beta\gamma\left(\omega+\omega^2\right) + \alpha\beta\left(\omega+\omega^2\right) + \gamma\alpha\left(\omega+\omega^2\right) &= 0\\
\beta^2 + \gamma^2 + \alpha^2 - \beta\gamma - \alpha\beta - \gamma\alpha &= 0
\end{aligned}$$

Where the last step uses $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

The fact $(*)$ was stated just as a fact, and it appears that no justification is necessary. However, I wanted to convince myself why this is true before using it. The diagram below shows the two different cases.



β-γ = ω(γ-α)



β-γ = ω²(γ-α)

For the first one, the triangle is equilateral if and only if point β is formed by translating point γ by $120°$ anticlockwise about α and then translating it by $\gamma - \alpha$. Using $z \mapsto \mathrm{e}^{i\theta}(z - c) + c$ for a rotation of θ anti clockwise about point c and that $\mathrm{e}^{i\frac{2\pi}{3}} = \omega$ gives us:

$$\beta = \omega\,(\gamma - \alpha) + \alpha + (\gamma - \alpha) \qquad \text{i.e.}$$
$$\beta - \gamma = \omega\,(\gamma - \alpha)$$

A similar argument can be used for the other triangle, but this time the rotation is 120° **clockwise** or equivalently 240° anti-clockwise, which is represented by multiplication by ω^2.

Having done this I then worked through this part of the question again, using the working shown above. I decided that I preferred my method, so presented that as the given "solution" here!

STEP III 2008 Q7 has another complex number question involving equilateral triangles.