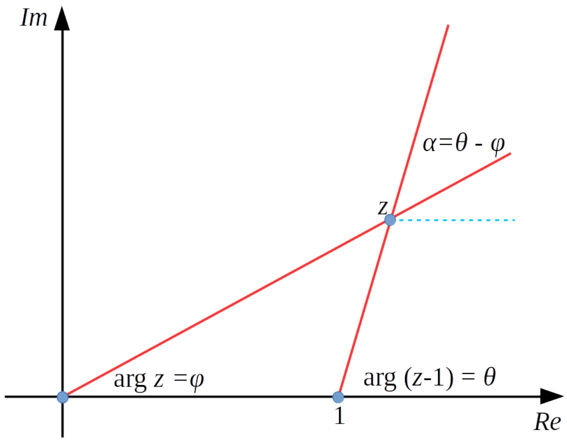
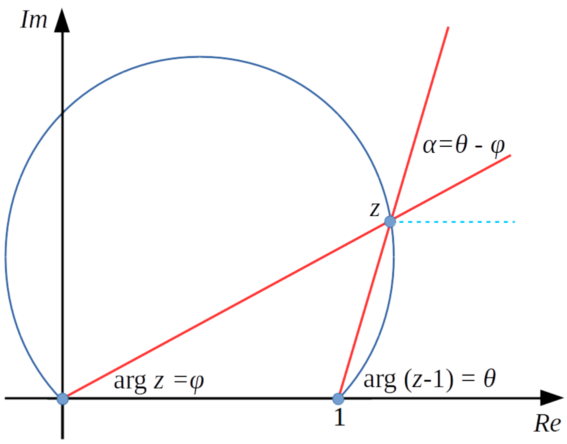
STEP Support Programme

STEP 2 Complex Numbers: Solutions

1 **(i)** Rewriting the given relationship gives $\arg\left(\dfrac{z-1}{z}\right) = \arg(z-1) - \arg(z) = \alpha$. We can then draw a picture as below:



The loci is therefore a section of the circle between $z = 0$ and $z = 1$. Since $0 < \alpha \leqslant \frac{1}{2}\pi$ the section will be above the real axis, and it will be the edge of the the major segment of the circle.



Alternatively

Let $z = x + \mathrm{i}y$. We then have:

$$\begin{aligned}
\arg\left(\frac{z-1}{z}\right) &= \alpha \\
\arg(z-1) - \arg(z) &= \alpha \\
\arg((x-1)+\mathrm{i}y) - \arg(x+\mathrm{i}y) &= \alpha \\
\tan\Big[\arg((x-1)+\mathrm{i}y) - \arg(x+\mathrm{i}y)\Big] &= \tan\alpha \\
\frac{\tan\Big(\arg((x-1)+\mathrm{i}y)\Big) - \tan\Big(\arg(x+\mathrm{i}y)\Big)}{1+\tan\Big(\arg((x-1)+\mathrm{i}y)\Big)\times\tan\Big(\arg(x+\mathrm{i}y)\Big)} &= \tan\alpha \\
\frac{\frac{y}{x-1} - \frac{y}{x}}{1 + \frac{y}{x-1}\times\frac{y}{x}} &= \tan\alpha \\
\frac{xy - y(x-1)}{x(x-1)+y^2} &= \tan\alpha \\
\frac{y}{x^2 - x + y^2} &= \tan\alpha
\end{aligned}$$

If we let $k = \dfrac{1}{\tan\alpha}$, then this becomes the equation $x^2 - x + y^2 - ky = 0$ i.e.

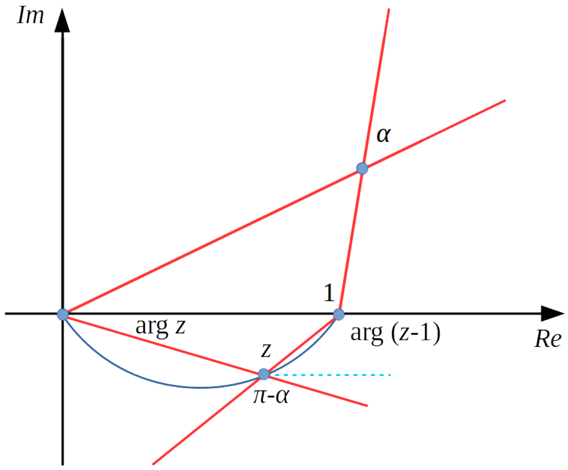
$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{k}{2}\right)^2 = \frac{k^2+1}{4}$$

Which is the equation of a circle. Bear in mind that you would still need to justify which part of the circle is actually the locus of z. You can do this by drawing a diagram, as before, or noting that if $\arg\left(\dfrac{z-1}{z}\right) = \alpha$ and $0 < \alpha \leqslant \frac{1}{2}\pi$ then the imaginary part of $\frac{z-1}{z}$ must be positive. Considering $\frac{z-1}{z}$ gives us:

$$\begin{aligned}
\frac{z-1}{z} &= \frac{(x-1)+\mathrm{i}y}{x+\mathrm{i}y} \\
&= \frac{[(x-1)+\mathrm{i}y][x-\mathrm{i}y]}{x^2+y^2} \\
&= \frac{x(x-1)+y^2+\mathrm{i}(xy-(x-1)y)}{x^2+y^2} \\
&= \frac{x^2+y^2-x}{x^2+y^2} + \mathrm{i}\frac{y}{x^2+y^2}
\end{aligned}$$

And so for the imaginary part to be positive we need $y > 0$, i.e. z must lie above the real axis.

(ii) If $0 < \alpha \leqslant \frac{1}{2}\pi$ then $-\pi < \alpha - \pi \leqslant -\frac{1}{2}\pi$. In this case we have a section of circle below the real axis, and the locus will be the edge of a minor segment of the circle, like the diagram below:



As before, you could use an algebraic approach if you prefer.

(iii) We have:

$$\left|\frac{z-1}{z}\right| = 1$$
$$\frac{|z-1|}{|z|} = 1$$
$$|z-1| = |z|$$

This means that the locus is all the points where the distance of z from the origin is the same as the distance of z from the point $(1, 0)$, i.e. it is the perpendicular bisector of the points $(0, 0)$ and $(1, 0)$. This will be the vertical line $x = 0.5$.

Alternatively

Let $z = x + \mathrm{i}y$. We have:

$$\begin{aligned}
\left|\frac{z-1}{z}\right| &= 1\\
\left|\frac{(x-1)+\mathrm{i}y}{x+\mathrm{i}y}\right| &= 1\\
\left|\frac{(x-1)x+y^2+\mathrm{i}(xy-y(x-1))}{x^2+y^2}\right| &= 1\\
\left|\frac{(x-1)x+y^2+\mathrm{i}y}{x^2+y^2}\right| &= 1\\
\frac{1}{x^2+y^2}\times\sqrt{\left(x^2+y^2-x\right)^2+y^2} &= 1\\
\sqrt{\left(x^2+y^2-x\right)^2+y^2} &= x^2+y^2\\
\left(x^2+y^2-x\right)^2+y^2 &= \left(x^2+y^2\right)^2\\
\cancel{\left(x^2+y^2\right)^2}+x^2-2x\left(x^2+y^2\right)+y^2 &= \cancel{\left(x^2+y^2\right)^2}\\
x^2+y^2 &= 2x\left(x^2+y^2\right)
\end{aligned}$$

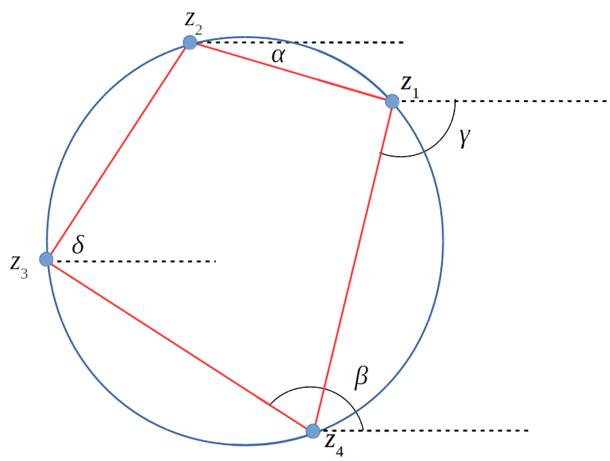
So either $x^2 + y^2 = 0$, which is not possible as this would imply that $z = 0$, or $2x = 1$ which gives the line $x = \frac{1}{2}$.

Although I have not done it here, the question did ask you to **sketch** the locus, so you would need to include a sketch!

For the last part, we have:

$$\arg w = \arg(z_1 - z_2) + \arg(z_3 - z_4) - \arg(z_4 - z_1) - \arg(z_2 - z_3)$$

Draw a diagram — which might look like this:



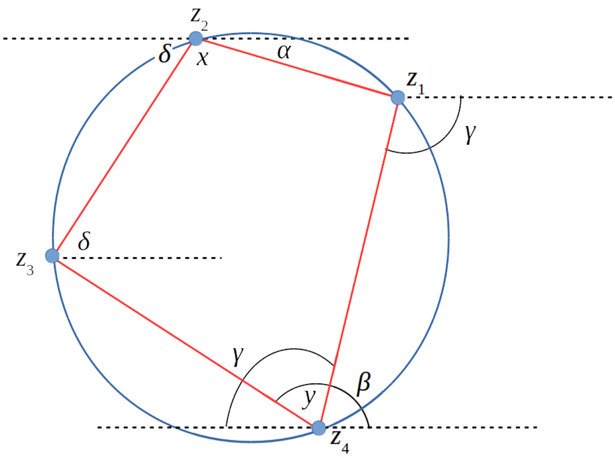
Then:

$$\arg(z_1 - z_2) = -\alpha$$
$$\arg(z_3 - z_4) = \beta$$
$$\arg(z_4 - z_1) = -\gamma$$
$$\arg(z_2 - z_3) = \delta$$

and so

$$\arg(z_1 - z_2) + \arg(z_3 - z_4) - \arg(z_4 - z_1) - \arg(z_2 - z_3) = -\alpha + \beta + \gamma - \delta$$

Add on some extra lines and angles (using the fact that alternate angles are equal) and we have:



We now have the equations:

$$\alpha + x + \delta = \pi \quad (1)$$
$$\beta + \gamma - y = \pi \quad (2)$$
$$x + y = \pi \quad (3)$$

Then (1) − (2) gives us:

$$\alpha + \delta - \beta - \gamma + x + y = 0$$

and using (3) we can write this as:

$$-\alpha + \beta + \gamma - \delta = \pi$$

Hence $\arg w = \pi$, and so w is on the real axis (and is real!).

2 Expanding the brackets gives us:

$$\begin{aligned}(\cos\theta + \mathrm{i}\sin\theta)(\cos\phi + \mathrm{i}\sin\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi + \mathrm{i}(\cos\theta\sin\phi + \sin\theta\cos\phi) \\ &= \cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi) \qquad (*)\end{aligned}$$

We know that the statement $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$ is true when $n = 1$. Assume it is true when $n = k$, so we have $(\cos\theta + \mathrm{i}\sin\theta)^k = \cos k\theta + \mathrm{i}\sin k\theta$. Now consider the case when $n = k + 1$.

$$\begin{aligned}(\cos\theta + \mathrm{i}\sin\theta)^{k+1} &= (\cos\theta + \mathrm{i}\sin\theta)^k \times (\cos\theta + \mathrm{i}\sin\theta) \\ &= (\cos k\theta + \mathrm{i}\sin k\theta)(\cos\theta + \mathrm{i}\sin\theta) \\ &= \cos(k\theta + \theta) + \mathrm{i}\sin(k\theta + \theta) \\ &= \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta\end{aligned}$$

where the second line uses the $n = k$ case and the third line uses $(*)$ with $\phi = k\theta$.

Hence if it is true for $n = k$ then it is true for $n = k + 1$, and as it is true for $n = 1$ it is true for all positive integers n.

Expanding $(5 - \mathrm{i})^2(1 + \mathrm{i})$ gives us:

$$\begin{aligned}(5-\mathrm{i})^2(1+\mathrm{i}) &= (24 - 10\mathrm{i})(1+\mathrm{i}) \\ &= 34 + 14\mathrm{i}\end{aligned}$$

Using what we know about arguments we have $2\arg(5 - \mathrm{i}) + \arg(1 + \mathrm{i}) = \arg(34 + 14\mathrm{i})$. Calculating the angles (and using a diagram to ensure we get the signs correct) gives us:

$$\begin{aligned}-2\tan^{-1}\left(\tfrac{1}{5}\right) + \tan^{-1}(1) &= \tan^{-1}\left(\tfrac{14}{34}\right) \\ -2\tan^{-1}\left(\tfrac{1}{5}\right) + \frac{\pi}{4} &= \tan^{-1}\left(\tfrac{7}{17}\right) \\ 2\tan^{-1}\left(\tfrac{1}{5}\right) + \tan^{-1}\left(\tfrac{7}{17}\right) &= \frac{\pi}{4}\end{aligned}$$

as required.

For the last part, you need to work out what complex numbers to use. Comparison with before suggests that it might be worth considering $\frac{\pi}{4} - \tan^{-1}\left(\tfrac{1}{20}\right) - 3\tan^{-1}\left(\tfrac{1}{4}\right)$. Consider $(1 + \mathrm{i})(20 - \mathrm{i})(4 - \mathrm{i})^3$ (by looking at the arguments in the required result)[1]. We have:

$$\begin{aligned}(1+\mathrm{i})(20-\mathrm{i})(4-\mathrm{i})^3 &= (21 + 19\mathrm{i})(64 - 48\mathrm{i} - 12 + \mathrm{i}) \\ &= (21 + 19\mathrm{i})(52 - 47\mathrm{i}) \\ &= 21 \times 52 + 19 \times 47 + \mathrm{i}(19 \times 52 - 21 \times 47) \\ &= 1985 + \mathrm{i}\end{aligned}$$

So we have $(1 + \mathrm{i})(20 - \mathrm{i})(4 - \mathrm{i})^3 = 1985 + \mathrm{i}$. Taking arguments gives us

$$\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{20}\right) - 3\tan^{-1}\left(\frac{1}{4}\right) = \tan^{-1}\left(\frac{1}{1985}\right)$$

and hence we have:

$$\tan^{-1}\left(\frac{1}{20}\right) + 3\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{1985}\right) = \frac{\pi}{4}$$

as required.

[1] There are different approaches to this question, using slightly different complex numbers such as $20 + \mathrm{i}$.

3 Substituting into w gives us:

$$\begin{aligned} w &= \frac{1+i(x+iy)}{i+(x+iy)} \\ &= \frac{1-y+ix}{x+i(1+y)} \\ &= \frac{\big(1-y+ix\big)\big(x-i(1+y)\big)}{\big(x+i(1+y)\big)\big(x-i(1+y)\big)} \\ &= \frac{x(1-\not{y})+x(1+\not{y})+i\left(x^2-(1-y)(1+y)\right)}{x^2+(1+y)^2} \\ &= \frac{2x+i\left(x^2+y^2-1\right)}{x^2+(1+y)^2} \end{aligned}$$

Therefore we have $u = \dfrac{2x}{x^2+(1+y)^2}$ and $v = \dfrac{x^2+y^2-1}{x^2+(1+y)^2}$.

This part of the question is called the "stem" of the question. The results that you have shown here can be assumed now throughout the rest of the question. Note that this only holds for "stem" results, you cannot assume that a result shown in part **(i)** holds in part **(ii)** etc.

(i) Setting $x = \tan(\theta/2)$[2] and $y = 0$ gives

$$\begin{aligned} u &= \frac{2\tan\left(\frac{\theta}{2}\right)}{\tan^2\left(\frac{\theta}{2}\right)+1} \\ &= \frac{2\tan\left(\frac{\theta}{2}\right)\times\cos^2\left(\frac{\theta}{2}\right)}{\left(\tan^2\left(\frac{\theta}{2}\right)+1\right)\times\cos^2\left(\frac{\theta}{2}\right)} \\ &= \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)+\cos^2\left(\frac{\theta}{2}\right)} \\ &= \frac{\sin\theta}{1} \\ &= \sin\theta \end{aligned}$$

and

$$\begin{aligned} v &= \frac{\tan^2\left(\frac{\theta}{2}\right)-1}{\tan^2\left(\frac{\theta}{2}\right)+1} \\ &= \frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)\times\cos^2\left(\frac{\theta}{2}\right)}{\left(\tan^2\left(\frac{\theta}{2}\right)+1\right)\times\cos^2\left(\frac{\theta}{2}\right)} \\ &= \frac{\sin^2\left(\frac{\theta}{2}\right)-\cos^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)+\cos^2\left(\frac{\theta}{2}\right)} \\ &= -\cos\theta \end{aligned}$$

This means that we have $u^2+v^2 = \sin^2\theta+\cos^2\theta = 1$. However, $\tan\phi$ is undefined for $\phi = \frac{\pi}{2}$, so we need to exclude $\theta = \pi$, i.e. the point $(\sin\pi, -\cos\pi) = (0,1)$ is excluded.

[2] Setting $x = \tan(\theta/2)$ means that taking values of θ in the range $-\pi < \theta < \pi$ will give x values in the range $-\infty < x < \infty$.

(ii) This case is almost identical to the previous one except that x is restricted to $-1 < x < 1$. This means we need to restrict θ so that $-1 < \tan\left(\frac{\theta}{2}\right) < 1$ i.e. $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Since $u = \sin\theta$ we have $-1 < u < 1$ and since $v = -\cos\theta$ we have $-1 < v < 0$. Hence the locus is the semi-circle which is part of $u^2 + v^2 = 1$ which lies (strictly) below the v-axis.

(iii) If we let $x = 0$ we get $u = 0$ and $v = \dfrac{y^2 - 1}{(1+y)^2} = \dfrac{y-1}{y+1}$ (using the results from the "stem" of the question). Hence we are restricted to the v-axis and as $-1 < y < 1$ we have $-\infty < v < 0$. Therefore the locus is the negative v-axis (i.e. the negative imaginary axis).

(iv) As this looks similar to part **(i)** we can try using $x = \tan\left(\frac{\theta}{2}\right)$ again. This gives:

$$u = \frac{2\tan\left(\frac{\theta}{2}\right)}{\tan^2\left(\frac{\theta}{2}\right) + 4}$$

This doesn't look promising as the denominator will not simplify nicely. Instead try using $x = 2\tan\left(\frac{\theta}{2}\right)$, which gives:

$$\begin{aligned}
u &= \frac{4\tan\left(\frac{\theta}{2}\right)}{4\tan^2\left(\frac{\theta}{2}\right) + 4} \\
&= \frac{\tan\left(\frac{\theta}{2}\right)}{\tan^2\left(\frac{\theta}{2}\right) + 1} \\
&= \frac{\left(\tan\left(\frac{\theta}{2}\right)\right) \times \cos^2\left(\frac{\theta}{2}\right)}{\left(\tan^2\left(\frac{\theta}{2}\right) + 1\right) \times \cos^2\left(\frac{\theta}{2}\right)} \\
&= \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\
&= \frac{1}{2}\sin\theta
\end{aligned}$$

We also get:

$$\begin{aligned}
v &= \frac{4\tan^2\left(\frac{\theta}{2}\right)}{4\tan^2\left(\frac{\theta}{2}\right) + 4} \\
&= \frac{\left(\tan^2\left(\frac{\theta}{2}\right)\right) \times \cos^2\left(\frac{\theta}{2}\right)}{\left(\tan^2\left(\frac{\theta}{2}\right) + 1\right) \times \cos^2\left(\frac{\theta}{2}\right)} \\
&= \sin^2\left(\frac{\theta}{2}\right) \\
&= \frac{1}{2}(1 - \cos\theta)
\end{aligned}$$

where the last step uses $\cos 2A = 1 - 2\sin^2 A$.

Like in part **(i)** we cannot have $\theta = \pi$, so the point $\left(\frac{1}{2}\sin\pi, \frac{1}{2}(1 - \cos\pi)\right) = (0, 1)$ is excluded.

The final stage is to describe this locus fully. As we have $\sin\theta$ and $\cos\theta$ involved it is likely to be something circle related. We have:

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= 1\\ (2u)^2 + (1-2v)^2 &= 1\\ 4u^2 + 4\left(\tfrac{1}{2} - v\right)^2 &= 1\\ u^2 + \left(\tfrac{1}{2} - v\right)^2 &= \tfrac{1}{4}\\ u^2 + \left(v - \tfrac{1}{2}\right)^2 &= \tfrac{1}{4}\end{aligned}$$

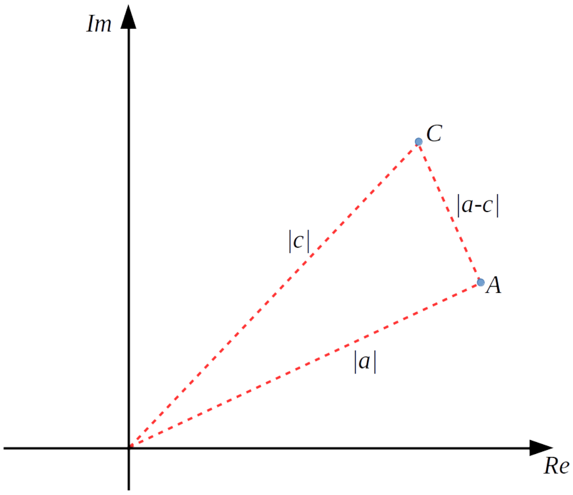
hence the locus is a circle radius $\frac{1}{2}$, centre $\left(0, \frac{1}{2}\right)$ with the point $(0, 1)$ excluded.

I haven't included any sketches in this solution, but a very good way of making your answers clear would be to sketch the loci (make sure the labels on your axes are correct, i.e. u and v on the real and imaginary axes). You can use empty circles to show the excluded points.

4 Using $|z|^2 = zz^*$ gives:

$$\begin{aligned}|a-c|^2 &= (a-c)(a^*-c^*)\\ &= aa^* + cc^* - ca^* - ac^*\end{aligned}$$

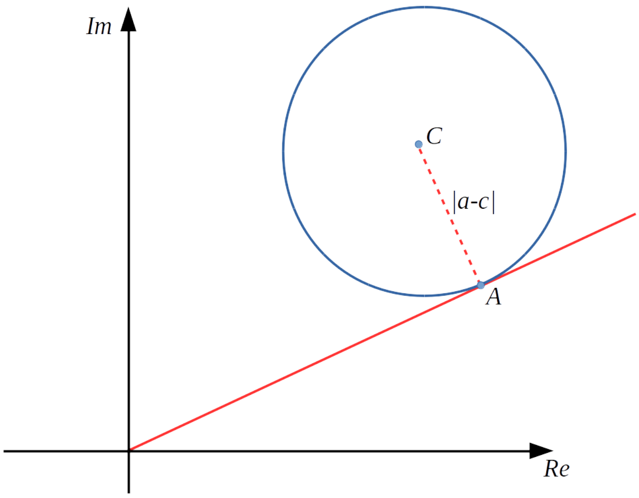
It is helpful to draw a diagram to show the points in the argand plane representing a and c:



Then the triangle has a right angle at A if and only if $|c|^2 = |a|^2 + |a-c|^2$, i.e.:

$$\begin{aligned}\iff \quad \cancel{cc^*} &= aa^* + aa^* + \cancel{cc^*} - ca^* - ac^*\\ \iff \quad 0 &= 2aa^* - ca^* - ac^*\\ \iff \quad 2aa^* &= ca^* + ac^*\end{aligned}$$

The circle and tangent could look something like:



The equation of the circle will be $|z-c|^2 = |a-c|^2$. This can be written as:

$$\begin{aligned}(z-c)(z^*-c^*) &= (a-c)(a^*-c^*)\\ zz^* + \cancel{cc^*} - cz^* - zc^* &= aa^* + \cancel{cc^*} - ca^* - ac^*\end{aligned}$$

Since the tangent is perpendicular to the circle we have $2aa^* = ca^* + ac^*$ or $2aa^* - ca^* - ac^* = 0$ and so we can write the equation of the circle as $zz^* - zc^* - cz^* + aa^* = 0$.

P will lie on this circle if and only if:

$$\begin{aligned} (ab)(ab)^* - (ab)c^* - c(ab)^* + aa^* &= 0 \qquad \text{i.e.} \\ aba^*b^* - abc^* - ca^*b^* + aa^* &= 0 \qquad (*) \end{aligned}$$

And P' will lie on this circle if and only if:

$$\begin{aligned} \left(\frac{a}{b^*}\right)\left(\frac{a}{b^*}\right)^* - \left(\frac{a}{b^*}\right)c^* - c\left(\frac{a}{b^*}\right)^* + aa^* &= 0 \qquad \text{i.e.} \\ \frac{aa^*}{bb^*} - \frac{ac^*}{b^*} - \frac{ca^*}{b} + aa^* &= 0 \qquad \text{multiply by } bb^*(\neq 0) \text{ to give} \\ aa^* - abc^* - ca^*b^* + aa^*bb^* &= 0 \end{aligned}$$

This last line is the exactly same condition as $(*)$, so point P is on the circle if and only if point P' is on the circle.

For the converse we cannot assume that OA is a tangent, so we cannot use the condition $2aa^* = ac^* + ca^*$. Going back to $|z - c|^2 = |a - c|^2$ gives:

$$\begin{aligned} (z - c)(z^* - c^*) &= (z - a)(z^* - a^*) \\ zz^* + \cancel{cc^*} - cz^* - zc^* &= aa^* + \cancel{cc^*} - ca^* - ac^* \end{aligned}$$

If both P and P' lie on this circle then we have:

$$\begin{aligned} (ab)(ab)^* - c(ab)^* - (ab)c^* &= aa^* - ca^* - ac^* \qquad \text{and} \qquad (4) \\ \left(\frac{a}{b^*}\right)\left(\frac{a}{b^*}\right)^* - c\left(\frac{a}{b^*}\right)^* - \left(\frac{a}{b^*}\right)c^* &= aa^* - ca^* - ac^* \qquad \text{i.e.} \qquad (5) \\ aa^* - ca^*b^* - abc^* &= (aa^* - ca^* - ac^*)\,bb^* \qquad (6) \end{aligned}$$

If we can show that $2aa^* - ca^* - ac^* = 0$ then we will have shown that $\angle OAC = 90^\circ$ and hence that OA is perpendicular to AC and so the line OA is a tangent to the circle.

If we take equation (4) and add aa^* to both sides, and take equation (6) and add aa^*bb^* we get the equations:

$$\begin{aligned} aa^*bb^* - ca^*b^* - abc^* + aa^* &= 2aa^* - ca^* - ac^* \qquad \text{and} \qquad (7) \\ aa^* - ca^*b^* - abc^* + aa^*bb^* &= (2aa^* - ca^* - ac^*)\,bb^* \qquad (8) \end{aligned}$$

The left hand side of equations (7) and (8) are the same, so if we consider (8) − (7) we get:

$$\begin{aligned} 0 &= (2aa^* - ca^* - ac^*)\,bb^* - (2aa^* - ca^* - ac^*) \\ 0 &= (2aa^* - ca^* - ac^*)\,(bb^* - 1) \end{aligned}$$

and since we are told that $bb^* \neq 1$ we have $2aa^* - ca^* - ac^* = 0$ and the line OA is a tangent to the circle.

Note that if $bb^* = 1$ then $b = \dfrac{1}{b^*}$. This means that $\dfrac{a}{b*} = ab$ and hence $P = P'$.