STEP Support Programme

STEP 2 Complex Numbers Topic Notes

Summation

To find $(a + \mathrm{i}b) + (c + \mathrm{i}d)$, add the real and imaginary parts separately to get $(a + c) + \mathrm{i}(b + d)$. This is equivalent to adding two vectors to get $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$. Subtraction is similar.

Multiplication

To find $(a + \mathrm{i}b)(c + \mathrm{i}d)$, multiply out the brackets as usual and then gather the real and imaginary parts, using $\mathrm{i}^2 = -1$.

Complex conjugate

The *conjugate* of $z = a + \mathrm{i}b$ is given by $z^* = a - \mathrm{i}b$. To simplify a complex number division multiply top and bottom by the complex conjugate, e.g:

$$\frac{3+\mathrm{i}}{1-2\mathrm{i}} = \frac{(3+\mathrm{i})(1+2\mathrm{i})}{(1-2\mathrm{i})(1+2\mathrm{i})} = \frac{3+\mathrm{i}+6\mathrm{i}-2}{1+4} = \frac{1+7\mathrm{i}}{5}$$

The roots of a polynomial with real coefficients are either real, or occur in complex conjugate pairs.

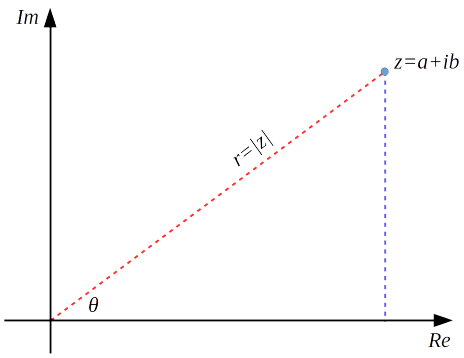
Note that the above statement holds for polynomials with **real** coefficients, if a polynomial has complex coefficients then not of all the roots will be real or occur in complex conjugate pairs.

Square roots

To find the square roots of $3 + 4\mathrm{i}$, let $(a + \mathrm{i}b)^2 = 3 + 4\mathrm{i}$, equate real and imaginary parts and solve for a and b.

Modulus and argument

A complex number $a + \mathrm{i}b$ **(Cartesian form)** can be represented by the point (a, b) in the Argand plane.



The modulus of the complex number is given by $|z|^2 = zz^* = a^2 + b^2$, and the argument is the angle the line connecting the origin to the point (a, b) makes to the x-axis measured anti-clockwise. The modulus is usually given in the range $-\pi < \theta \leqslant \pi$ (or sometimes $0 \leqslant \theta < 2\pi$).

Using trigonometry we can then write $z = r\cos\theta + \mathrm{i}r\sin\theta = r(\cos\theta + \mathrm{i}\sin\theta)$, known as **modulus-argument** or **polar** form.

Multiplication and division in polar form

If we have $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$ then:

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos\theta_1 + \mathrm{i}\sin\theta_1)(\cos\theta_2 + \mathrm{i}\sin\theta_2) \\ &= r_1 r_1 \big(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + \mathrm{i}(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\big) \\ &= r_1 r_1 \big(\cos(\theta_1+\theta_2) + \mathrm{i}\sin(\theta_1+\theta_2)\big) \end{aligned}$$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1}{r_2} \times \frac{\cos\theta_1 + \mathrm{i}\sin\theta_1}{\cos\theta_2 + \mathrm{i}\sin\theta_2} \\ &= \frac{r_1}{r_2} \times \frac{(\cos\theta_1 + \mathrm{i}\sin\theta_1)(\cos\theta_2 - \mathrm{i}\sin\theta_2)}{(\cos\theta_2 + \mathrm{i}\sin\theta_2)(\cos\theta_2 - \mathrm{i}\sin\theta_2)} \\ &= \frac{r_1}{r_2} \times \frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + \mathrm{i}(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2} \\ &= \frac{r_1}{r_2}\big(\cos(\theta_1-\theta_2) + \mathrm{i}\sin(\theta_1-\theta_2)\big) \end{aligned}$$

I.e. when multiplying two numbers the moduli are multiplied and the arguments added together and when dividing one complex number by another, the modulus of the first is divided by the modulus of the second and the argument of the second is subtracted from the argument of the first.

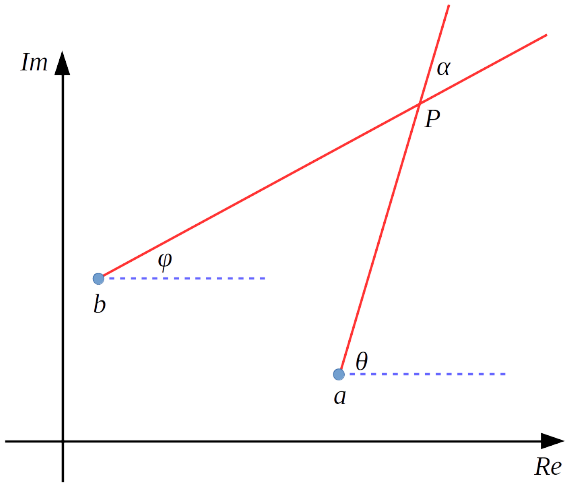
Loci

Remember that:

$$|z_1 z_2| = |z_1| \times |z_2| \qquad \text{and} \qquad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$
$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad \text{and} \qquad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

- $|z| = r$ is a circle radius r centre the origin
- $|z - c| = r$ is a circle radius r centre c
- $|z-a| = |z-b|$ means that z is equidistant from both a and b, i.e. the locus is the perpendicular bisector of the line connecting a and b
- $\arg z = \theta$ is the half line from the origin making angle θ from the positive x-axis (measured anti-clockwise)
- $\arg(z - c) = \theta$ is the half line from point c making angle θ with the horizontal

- $\arg\left(\dfrac{z-a}{z-b}\right) = \alpha$ represents a partial circle with ends points at a and b. This can be seen by writing the equation as $\arg(z-a) - \arg(z-b) = \alpha$, and then letting $\arg(z-a) = \theta$ and $\arg(z-b) = \phi$. We then have a picture like the one below:



Then as θ and ϕ vary they do so in such a way as to keep $\theta - \phi = \alpha$, and since "angles subtended from an arc are equal" P traces out part of the circumference of a circle.

For other loci, it might be helpful to write $z = x + \mathrm{i}y$ and then find the equation of the locus in Cartesian form.

Transformations

- $z \mapsto z + a$ is a translation by complex number a (which can be thought of as a vector)
- $z \mapsto \lambda z$, where lambda is real, is an enlargement scale factor λ centre the origin
- $z \mapsto z \times (\cos\theta + \mathrm{i}\sin\theta)$ is a rotation about the origin by angle θ anticlockwise
- $z \mapsto (z-c) \times (\cos\theta + \mathrm{i}\sin\theta) + c$ is a rotation about the point c by angle θ anticlockwise (this is a combination of a translation so that point c is at the origin, followed by a rotation and finally a translation back so that c returns to its original place)