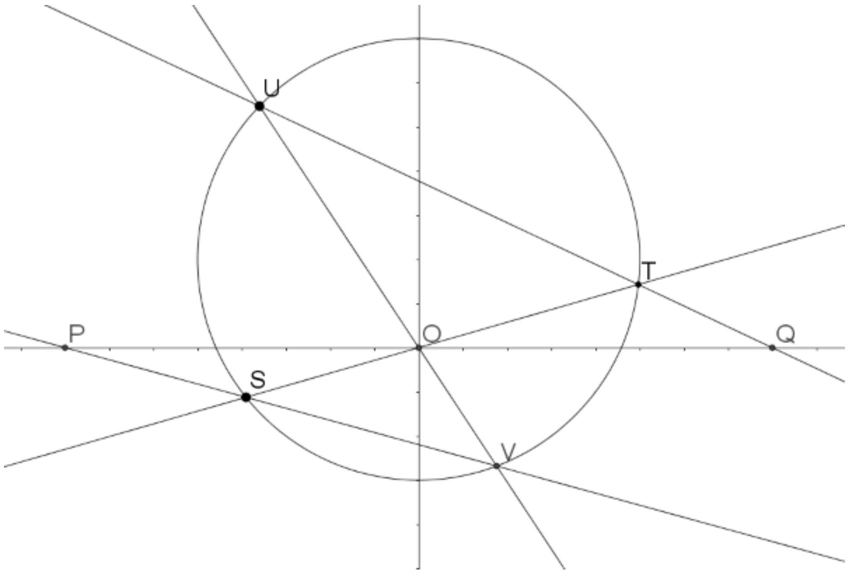
STEP Support Programme

STEP 3 Polar Coordinates and other Coordinate Geometry: Solutions

These are not fully worked solutions — you need to fill in some gaps. It is a good idea to look at the "Hints" document before this one.

1 First, we need a nice clear diagram. This one includes the circle mentioned in the second part:



Note that the lines ST and UV pass through the origin, as S and T both lie on the line $y = mx$ and U and V both lie on the line $y = nx$.

Gradient of SV : $\frac{ms-nv}{s-v}$

Equation of SV : $y - ms = \frac{ms-nv}{s-v}(x - s)$

Setting $y = 0$ and rearranging to find x gives $p = \frac{(m-n)sv}{ms-nv}$ as required.

Similarly, $q = \frac{(m-n)ut}{mt-nu}$.

S and T are the points of intersection of $y = mx$ with $x^2 + (y - c)^2 = r^2$. Solving simultaneously,

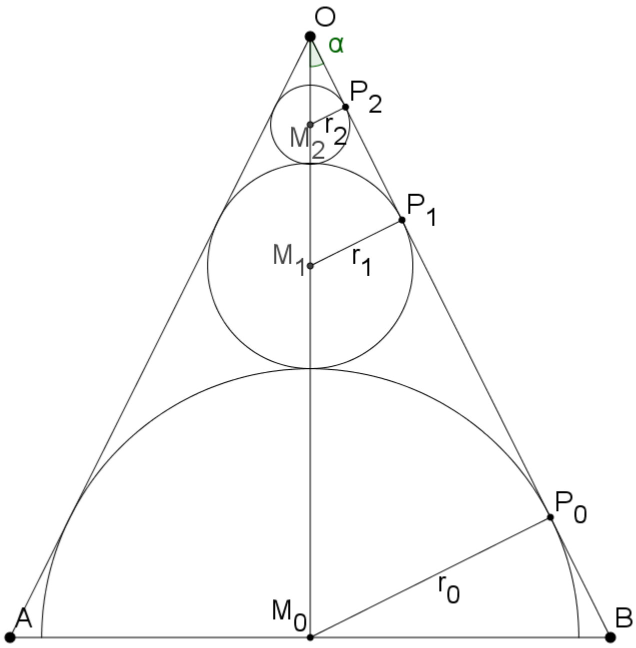
$$x^2 + (mx - c)^2 = r^2$$
$$\Rightarrow x^2 + m^2x^2 - 2mxc + c^2 - r^2 = 0$$
$$\Rightarrow x^2 - \frac{2mc}{1+m^2}x + \frac{c^2 - r^2}{1+m^2} = 0$$

s and t are the roots of this equation. $(x - s)(x - t) = x^2 - (s + t)x + st$, so $s + t = \frac{2mc}{1+m^2}$ and $st = \frac{c^2-r^2}{1+m^2}$.

As U and V are the points of intersection of the line $y = nx$ with the circle we have $u + v = \frac{2nc}{1+n^2}$ and $uv = \frac{c^2-r^2}{1+n^2}$.

$$\begin{aligned}
p + q &= \frac{(m-n)sv}{ms-nv} + \frac{(m-n)ut}{mt-nu} \\
&= \frac{(m-n)}{(ms-nv)(mt-nu)}\Big((mt-nu)sv + (ms-nv)ut\Big) \\
&= \frac{(m-n)}{(ms-nv)(mt-nu)}\Big(mvst - nsuv + must - ntuv\Big) \\
&= \frac{(m-n)}{(ms-nv)(mt-nu)}\Big(m(u+v)st - n(s+t)uv\Big) \\
&= \frac{(m-n)}{(ms-nv)(mt-nu)}\left(m \times \frac{2nc}{1+n^2} \times \frac{c^2-r^2}{1+m^2} - n \times \frac{2mc}{1+m^2} \times \frac{c^2-r^2}{1+n^2}\right) \\
&= \frac{(m-n)}{(ms-nv)(mt-nu)}\left(\frac{2mnc(c^2-r^2)}{(1+n^2)(1+m^2)} - \frac{2nmc(c^2-r^2)}{(1+m^2)(1+n^2)}\right) \\
&= 0 \quad \text{as required.}
\end{aligned}$$

2 Start with a diagram:



The centre of circle C_n is at M_n.

For any n, $|OM_n|\sin\alpha = r_n$. Note that $|OM_n| - r_n - r_{n+1} = |OM_{n+1}|$. Substituting for $|OM_{n+1}|$ and $|OM_n|$ and rearranging gives:

$$\begin{aligned}\frac{r_n}{\sin\alpha} - r_n - r_{n+1} &= \frac{r_{n+1}}{\sin\alpha}\\ r_n - r_n\sin\alpha &= r_{n+1} + r_{n+1}\sin\alpha\\ r_{n+1}(1+\sin\alpha) &= r_n(1-\sin\alpha)\\ \frac{r_{n+1}}{r_n} &= \frac{1-\sin\alpha}{1+\sin\alpha}\end{aligned}$$

as required. This might be slightly more usefully written as $r_{n+1} = \left(\dfrac{1-\sin\alpha}{1+\sin\alpha}\right) r_n$.

$$\text{Area of } C_0 = \frac{\pi r_0^2}{2}$$

$$\text{Area of } C_1 = \pi r_1^2 = \pi r_0^2\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^2$$

Continuing gives us a geometric series for the sum of the areas of the circles, with first term

$$\pi r_0^2\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^2$$

and common ratio

$$\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^2$$

Using the formula for the sum to infinity of a geometric series gives the sum of the areas of the circles as:

$$\begin{aligned}\frac{a}{1-r} &= \pi r_0^2 \left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^2 \times \frac{1}{1-\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^2} \\ &= \pi r_0^2 \frac{(1-\sin\alpha)^2}{(1+\sin\alpha)^2-(1-\sin\alpha)^2} \\ &= \pi r_0^2 \frac{(1-\sin\alpha)^2}{4\sin\alpha}\end{aligned}$$

Adding in the area of the semi-circle C_0 gives:

$$\begin{aligned}S &= \pi r_0^2 \frac{(1-\sin\alpha)^2}{4\sin\alpha} + \frac{\pi}{2}r_0^2 \\ &= \pi r_0^2 \left(\frac{(1-\sin\alpha)^2}{4\sin\alpha} + \frac{1}{2}\right) \\ &= \pi r_0^2 \left(\frac{1-\cancel{2\sin\alpha}+\sin^2\alpha+\cancel{2\sin\alpha}}{4\sin\alpha}\right) \\ &= \frac{1+\sin^2\alpha}{4\sin\alpha}\pi r_0^2\end{aligned}$$

as required.

$$\begin{aligned}\text{Area of triangle} &= |OM_0||M_0B| \\ T &= \frac{r_0}{\sin\alpha}\frac{r_0}{\cos\alpha}\end{aligned}$$

Thus the ratio of S to T is given by:

$$\begin{aligned}\frac{S}{T} &= \frac{1+\sin^2\alpha}{4\sin\alpha}\pi\sin\alpha\cos\alpha \\ &= \frac{\pi}{4}\cos\alpha(1+\sin^2\alpha)\end{aligned}$$

Differentiating with respect to α gives

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{S}{T}\right) = \frac{\pi}{4}\left(\cos\alpha\times 2\sin\alpha\cos\alpha - \sin\alpha(1+\sin^2\alpha)\right) .$$

Setting this equal to zero we find the maximum occurs when

$$\begin{aligned}\cos\alpha\times 2\sin\alpha\cos\alpha - \sin\alpha(1+\sin^2\alpha) &= 0 \\ \sin\alpha\left(2\cos^2\alpha - 1 - \sin^2\alpha\right) &= 0 \\ \sin\alpha\left(3\cos^2\alpha - 2\right) &= 0\end{aligned}$$

So either $\sin\alpha = 0$, which has no solutions in the range $0<\alpha<\frac{\pi}{2}$ or $\cos\alpha = \pm\sqrt{\frac{2}{3}}$. If $0<\alpha<\frac{\pi}{2}$ then we must have $\cos\alpha = \sqrt{\frac{2}{3}}$ and substituting gives $\frac{S}{T} = \frac{\pi}{4}\times\sqrt{\frac{2}{3}}\times\frac{4}{3} > \sqrt{\frac{2}{3}} = \sqrt{\frac{16}{24}} > \sqrt{\frac{16}{25}} = \frac{4}{5}$. Hence for this value of α we have $S > \frac{4}{5}T$.

3 **(i)** At a point of inflection we have $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$[1]. Start by differentiating once to get:

$$\begin{aligned} 3x^2 + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 3y + 3x\frac{\mathrm{d}y}{\mathrm{d}x} \\ \implies \frac{\mathrm{d}y}{\mathrm{d}x}(y^2 - x) &= y - x^2 \end{aligned}$$

Differentiating again gives:

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2}(y^2 - x) + \frac{\mathrm{d}y}{\mathrm{d}x}\left(2y\frac{\mathrm{d}y}{\mathrm{d}x} - 1\right) &= \frac{\mathrm{d}y}{\mathrm{d}x} - 2x \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}(y^2 - x) + 2y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2x &= 0 \end{aligned}$$

Then substituting $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y - x^2}{y^2 - x}$ gives:

$$\begin{aligned} \left(\frac{y - x^2}{y^2 - x}\right)^2 \times y - \frac{y - x^2}{y^2 - x} + x &= 0 \\ y(y - x^2)^2 - (y - x^2)(y^2 - x) + x(y^2 - x)^2 &= 0 \\ y(y^2 - 2yx^2 + x^4) - (y^3 - x^2y^2 - xy + x^3) + x(y^4 - 2y^2x + x^2) &= 0 \\ \cancel{y^3} - 2y^2x^2 + yx^4 - \cancel{y^3} + x^2y^2 + xy - \cancel{x^3} + xy^4 - 2y^2x^2 + \cancel{x^3} &= 0 \\ yx^4 + xy^4 + xy - 3x^2y^2 &= 0 \\ xy\left(x^3 + y^3\right) + xy - 3x^2y^2 &= 0 \\ xy \times 3xy + xy - 3x^2y^2 &= 0 \\ xy &= 0 \end{aligned}$$

So either $x = 0$ or $y = 0$, but from $x^3 + y^3 = 3xy$, if one of x and y is equal to zero then the other one is. So the only point where $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ is the origin, which we have been told is not a point of inflection. Hence there are no points of inflection.

(ii) In polar coordinates the area of the loop is given by:

$$\tfrac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \,\mathrm{d}\theta$$

Using $x = r\cos\theta$ and $y = r\sin\theta$ in the equation for the curve gives:

$$\begin{aligned} x^3 + y^3 &= 3xy \\ r^3\cos^3\theta + r^3\sin^3\theta &= 3r^2\cos\theta\sin\theta \\ \implies r &= \frac{3\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta} \quad \textit{since } r > 0 \end{aligned}$$

[1]Note that point of inflection $\implies \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ but $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \not\Longrightarrow$ point of inflection. This question asks us to show there are no points of inflection so showing that there are no points where $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ is sufficient, but if we were asked to show that there were points of inflection then showing that there are points where $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ is not enough.

The area of the loop is therefore:

$$\begin{aligned}\tfrac{1}{2}\int_0^{\frac{\pi}{2}}\left(\frac{3\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}\right)^2\mathrm{d}\theta &= \frac{9}{2}\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta\sin^2\theta}{\left(\cos^3\theta+\sin^3\theta\right)^2}\,\mathrm{d}\theta\\ &= \frac{9}{2}\int_0^{\frac{\pi}{2}}\frac{\sec^2\theta\tan^2\theta}{\left(1+\tan^3\theta\right)^2}\,\mathrm{d}\theta\end{aligned}$$

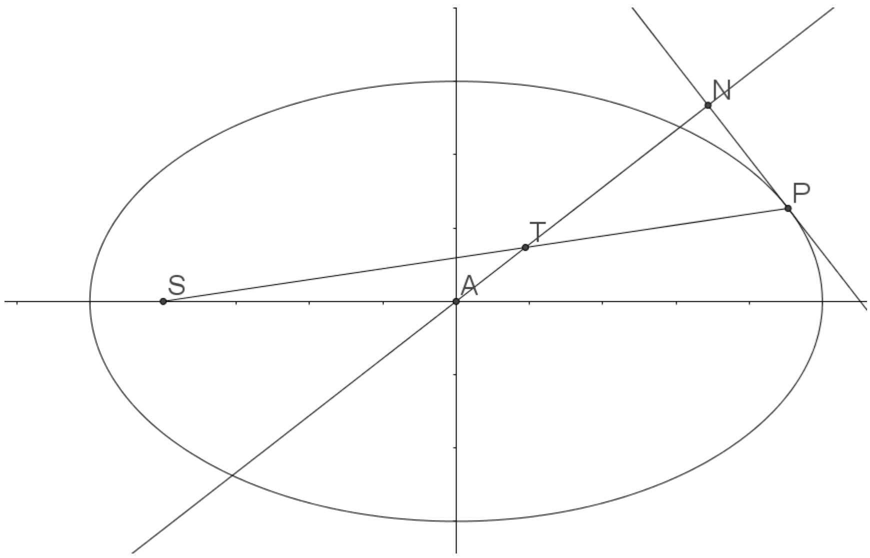
Where the last integral is found by dividing the numerator and denominator by $\tan^6\theta$. Using the substitution $\tan\theta = u$ gives:

$$\begin{aligned}\frac{9}{2}\int_0^{\frac{\pi}{2}}\frac{\sec^2\theta\tan^2\theta}{\left(1+\tan^3\theta\right)^2}\,\mathrm{d}\theta &= \frac{9}{2}\int_0^{\infty}\frac{\sec^2\theta\, u^2}{\left(1+u^3\right)^2}\times\frac{1}{\sec^2\theta}\,\mathrm{d}u\\ &= \frac{9}{2}\int_0^{\infty}\frac{u^2}{\left(1+u^3\right)^2}\,\mathrm{d}u\\ &= \frac{9}{2}\int_1^{\infty}\frac{u^2}{t^2}\times\frac{1}{3u^2}\,\mathrm{d}t\\ &= \frac{3}{2}\int_1^{\infty}t^{-2}\,\mathrm{d}t\\ &= \frac{3}{2}\Big[-t^{-1}\Big]_1^{\infty}\\ &= \frac{3}{2}\end{aligned}$$

The second substitution used here is $t = 1 + u^3$.

You might like to use Desmos to sketch the graph described by the equation given in the stem of the question.

4 As always, a diagram helps to set the scene:



The ellipse cuts the x axis at $(\pm a, 0)$ and the y axis at $(0, \pm b)$.

Differentiating

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

with respect to x and rearranging gives:

$$\begin{aligned}\frac{2x}{a^2} + \frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2x}{a^2}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{xb^2}{a^2y}\end{aligned}$$

So at P, the gradient is $-\frac{b\cos\theta}{a\sin\theta}$

So the gradient of ON is $\frac{a\sin\theta}{b\cos\theta}$

So ON has equation $y = \frac{a\sin\theta}{b\cos\theta}x$

Next, let us find the equation of the line joining S and P. It has gradient $\frac{b\sin\theta}{ea+a\cos\theta}$ and passes through $(-ea, 0)$ so its equation is given by

$$y = \frac{b\sin\theta}{ea + a\cos\theta}(x + ea)$$

We want to find the y coordinate at the intersection of SP and ON. Rearranging the equations of the lines to give x in terms of y, and equating, gives:

$$
\begin{aligned}
&& \left(\frac{ea + a\cos\theta}{b\sin\theta}\right)y - ea &= \left(\frac{b\cos\theta}{a\sin\theta}\right)y \\
\implies && y\left(\frac{ea + a\cos\theta}{b\sin\theta} - \frac{b\cos\theta}{a\sin\theta}\right) &= ea \\
\implies && y\left(\frac{a^2(e+\cos\theta) - b^2\cos\theta}{ab}\right) &= ea\sin\theta \\
\implies && y(a^2e + (a^2 - b^2)\cos\theta) &= ea^2b\sin\theta \\
\implies && y(a^2e + (a^2 - (a^2(1-e^2))\cos\theta) &= ea^2b\sin\theta \\
\implies && y(a^2e(1+e\cos\theta) &= ea^2b\sin\theta \\
\implies && y &= \frac{b\sin\theta}{1+e\cos\theta}
\end{aligned}
$$

As the point T lies on ON, we can substitute into the equation for ON to get the x coordinate of T: $\dfrac{b\sin\theta}{1+e\cos\theta} = \dfrac{a\sin\theta}{b\cos\theta}x$ so $x = \dfrac{b^2\cos\theta}{a+ae\cos\theta}$

The circle with centre S and radius a is the set of points (x,y) satisfying $(x+ea)^2+y^2 = a^2$. For convenience, let $K = 1+e\cos\theta$[2]. Substituting our values for x and y at the point T on the LHS gives:

$$
\begin{aligned}
&\left(\frac{b^2\cos\theta}{aK} + ea\right)^2 + \left(\frac{b\sin\theta}{K}\right)^2 \\
&= \frac{b^4\cos^2\theta}{a^2K^2} + \frac{2eb^2\cos\theta}{K} + e^2a^2 + \frac{b^2\sin^2\theta}{K^2} \\
&= \frac{(1-e^2)b^2\cos^2\theta}{K^2} + \frac{2eb^2\cos\theta}{K} + e^2a^2 + \frac{b^2\sin^2\theta}{K^2} \qquad \text{since } \frac{b^2}{a^2} = 1-e^2 \\
&= \frac{b^2}{K^2}\cos^2\theta - \frac{b^2}{K^2}e^2\cos^2\theta + \frac{2eb^2\cos\theta}{K} + e^2a^2 + \frac{b^2\sin^2\theta}{K^2} \\
&= \frac{b^2}{K^2}(1-e^2\cos^2\theta) + \frac{2eb^2\cos\theta}{K} + e^2a^2 \\
&= \frac{b^2}{K^2}(1-e\cos\theta)(1+e\cos\theta) + \frac{2eb^2\cos\theta}{K} + e^2a^2 \\
&= \frac{b^2}{K}(1-e\cos\theta) + \frac{2eb^2\cos\theta}{K} + e^2a^2 \\
&= \frac{b^2}{K}(1+e\cos\theta) + e^2a^2 \\
&= b^2 + e^2a^2 \\
&= a^2 - e^2a^2 + e^2a^2 \\
&= a^2
\end{aligned}
$$

Thus, the coordinates of T satisfy $(x+ea)^2+y^2 = a^2$ and so T lies on the circle.

[2]It is important to bear in mind that K is **not** constant.

5 Note that $r^2 = x^2 + y^2$, and $\tan\theta = \frac{y}{x}$. Differentiating the second expression with respect to t gives:

$$\sec^2\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{x^2}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)$$

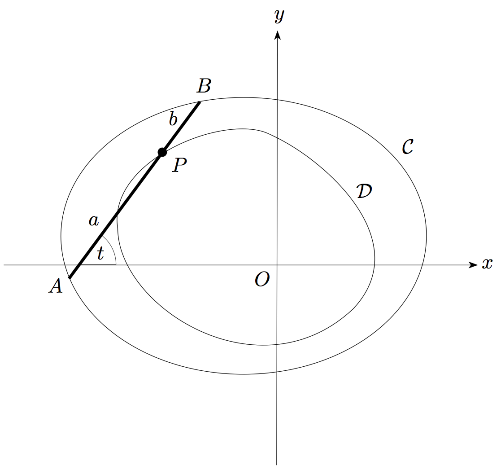
Hence,

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x^2}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\\ &= \frac{1}{x^2+y^2}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\end{aligned}$$

Then,

$$\begin{aligned}\frac{1}{2}\int r^2\,\mathrm{d}\theta &= \frac{1}{2}\int (x^2+y^2)\mathrm{d}\theta\\ &= \frac{1}{2}\int (x^2+y^2)\frac{\mathrm{d}\theta}{\mathrm{d}t}\mathrm{d}t\\ &= \frac{1}{2}\int (x^2+y^2)\frac{1}{x^2+y^2}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t\\ &= \frac{1}{2}\int \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t \qquad (*)\end{aligned}$$

as required.



If $P = (x, y)$, then $A = (x - a\cos t, y - a\sin t)$ and $B = (x + b\cos t, y + b\sin t)$.
Using $(*)$, $[A] = \frac{1}{2}\int\left(X\frac{\mathrm{d}Y}{\mathrm{d}t} - Y\frac{\mathrm{d}X}{\mathrm{d}t}\right)\mathrm{d}t$, where $X = x - a\cos t$ and $Y = y - a\sin t$. Note that $[P] = \frac{1}{2}\int\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t$.

$$\begin{aligned}
[A] &= \frac{1}{2}\int_0^{2\pi} \left[(x - a\cos t)\left(\frac{\mathrm{d}y}{\mathrm{d}t} - a\cos t\right) - (y - a\sin t)\left(\frac{\mathrm{d}x}{\mathrm{d}t} + a\sin t\right)\right] \mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} \left[x\frac{\mathrm{d}y}{\mathrm{d}t} - a\cos t\frac{\mathrm{d}y}{\mathrm{d}t} - xa\cos t + a^2\cos^2 t - y\frac{\mathrm{d}x}{\mathrm{d}t} + a\sin t\frac{\mathrm{d}x}{\mathrm{d}t} - ay\sin t + a^2\sin^2 t\right] \mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t - \frac{a}{2}\int_0^{2\pi}\left(\cos t\frac{\mathrm{d}y}{\mathrm{d}t} + x\cos t - \sin t\frac{\mathrm{d}x}{\mathrm{d}t} + y\sin t\right)\mathrm{d}t \\
&\qquad + \frac{a^2}{2}\int_0^{2\pi}(\cos^2 t + \sin^2 t)\,\mathrm{d}t \\
&= [P] - \frac{a}{2}\int_0^{2\pi}\left[\left(\frac{\mathrm{d}y}{\mathrm{d}t} + x\right)\cos t + \left(y - \frac{\mathrm{d}x}{\mathrm{d}t}\right)\sin t\right]\mathrm{d}t + \frac{a^2}{2}\int_0^{2\pi} 1\,\mathrm{d}t \\
&= [P] - af + \pi a^2
\end{aligned}$$

B has coordinates as A but with $a = -b$, so

$$[B] = [P] + bf + \pi b^2$$

Since $[A] = [B]$, $\pi a^2 - af = \pi b^2 + bf \implies (a+b)f = \pi(a^2 - b^2) \implies f = \pi(a-b)$.

The desired area is $[A] - [P] = \pi a^2 - a\big(\pi(a-b)\big) = \pi ab$ as required.