1 2009 S3 Q1
The points $S$, $T$, $U$ and $V$ have coordinates $(s, ms)$, $(t, mt)$, $(u, nu)$ and $(v, nv)$, respectively. The lines $SV$ and $UT$ meet the line $y = 0$ at the points with coordinates $(p, 0)$ and $(q, 0)$, respectively. Show that

$$p = \frac{(m - n)sv}{ms - nv},$$

and write down a similar expression for $q$.

Given that $S$ and $T$ lie on the circle $x^2 + (y - c)^2 = r^2$, find a quadratic equation satisfied by $s$ and by $t$, and hence determine $st$ and $s + t$ in terms of $m$, $c$ and $r$.

Given that $S$, $T$, $U$ and $V$ lie on the above circle, show that $p + q = 0$.

2 2004 S3 Q4
The triangle $OAB$ is isosceles, with $OA = OB$ and angle $AOB = 2\alpha$ where $0 < \alpha < \frac{\pi}{2}$. The semi-circle $C_0$ has its centre at the midpoint of the base $AB$ of the triangle, and the sides $OA$ and $OB$ of the triangle are both tangent to the semi-circle. $C_1, C_2, C_3, \ldots$ are circles such that $C_n$ is tangent to $C_{n-1}$ and to sides $OA$ and $OB$ of the triangle.

Let $r_n$ be the radius of $C_n$. Show that

$$\frac{r_{n+1}}{r_n} = \frac{1 - \sin \alpha}{1 + \sin \alpha}.$$

Let $S$ be the total area of the semi-circle $C_0$ and the circles $C_1$, $C_2$, $C_3$, \ldots. Show that

$$S = \frac{1 + \sin^2 \alpha}{4 \sin \alpha} \pi r_0^2.$$

Show that there are values of $\alpha$ for which $S$ is more than four fifths of the area of triangle $OAB$. 
3  **1993 S3 Q2**

The curve $C$ has the equation $x^3 + y^3 = 3xy$.

(i) Show that there is no point of inflection on $C$. You may assume that the origin is not a point of inflection.

(ii) The part of $C$ which lies in the first quadrant is a closed loop touching the axes at the origin. By converting to polar coordinates, or otherwise, evaluate the area of this loop.

4  **2008 S3 Q3**

The point $P(a \cos \theta, b \sin \theta)$, where $a > b > 0$, lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. $$

The point $S(-ea, 0)$, where $b^2 = a^2(1 - e^2)$, is a focus of the ellipse. The point $N$ is the foot of the perpendicular from the origin, $O$, to the tangent to the ellipse at $P$. The lines $SP$ and $ON$ intersect at $T$. Show that the $y$-coordinate of $T$ is

$$\frac{b \sin \theta}{1 + e \cos \theta}. $$

Show that $T$ lies on the circle with centre $S$ and radius $a.$
5 2011 S3 Q5

A movable point $P$ has cartesian coordinates $(x, y)$, where $x$ and $y$ are functions of $t$. The polar coordinates of $P$ with respect to the origin $O$ are $r$ and $\theta$. Starting with the expression

$$\frac{1}{2} \int r^2 \, d\theta$$

for the area swept out by $OP$, obtain the equivalent expression

$$\frac{1}{2} \int \left( \frac{dy}{dt} - \frac{dx}{dt} \right) dt. \quad (\star)$$

The ends of a thin straight rod $AB$ lie on a closed convex curve $C$. The point $P$ on the rod is a fixed distance $a$ from $A$ and a fixed distance $b$ from $B$. The angle between $AB$ and the positive $x$ direction is $t$. As $A$ and $B$ move anticlockwise round $C$, the angle $t$ increases from $0$ to $2\pi$ and $P$ traces a closed convex curve $D$ inside $C$, with the origin $O$ lying inside $D$, as shown in the diagram.

Let $(x, y)$ be the coordinates of $P$. Write down the coordinates of $A$ and $B$ in terms of $a$, $b$, $x$, $y$ and $t$.

The areas swept out by $OA$, $OB$ and $OP$ are denoted by $[A]$, $[B]$ and $[P]$, respectively. Show, using $(\star)$, that

$$[A] = [P] + \pi a^2 - af$$

where

$$f = \frac{1}{2} \int_0^{2\pi} \left( \left( x + \frac{dy}{dt} \right) \cos t + \left( y - \frac{dx}{dt} \right) \sin t \right) dt.$$

Obtain a corresponding expression for $[B]$ involving $b$. Hence show that the area between the curves $C$ and $D$ is $\pi ab$. 

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STEP 3 Polar: Questions