STEP Support Programme

STEP 3 Polar Coordinates and other Coordinate Geometry: Questions

1 2009 S3 Q1

The points S, T, U and V have coordinates (s, ms), (t, mt), (u, nu) and (v, nv), respectively. The lines SV and UT meet the line $y = 0$ at the points with coordinates $(p, 0)$ and $(q, 0)$, respectively. Show that

$$p = \frac{(m-n)sv}{ms - nv}\,,$$

and write down a similar expression for q.

Given that S and T lie on the circle $x^2 + (y-c)^2 = r^2$, find a quadratic equation satisfied by s and by t, and hence determine st and $s+t$ in terms of m, c and r.

Given that S, T, U and V lie on the above circle, show that $p + q = 0$.

2 2004 S3 Q4

The triangle OAB is isosceles, with $OA = OB$ and angle $AOB = 2\alpha$ where $0 < \alpha < \frac{\pi}{2}$. The semi-circle C_0 has its centre at the midpoint of the base AB of the triangle, and the sides OA and OB of the triangle are both tangent to the semi-circle. $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \ldots$ are circles such that C_n is tangent to C_{n-1} and to sides OA and OB of the triangle.

Let r_n be the radius of C_n. Show that

$$\frac{r_{n+1}}{r_n} = \frac{1 - \sin\alpha}{1 + \sin\alpha}\,.$$

Let S be the total area of the semi-circle C_0 and the circles C_1, C_2, C_3, $\ldots$. Show that

$$S = \frac{1 + \sin^2\alpha}{4\sin\alpha}\,\pi r_0^2\,.$$

Show that there are values of α for which S is more than four fifths of the area of triangle OAB.

3 1993 S3 Q2

The curve C has the equation $x^3 + y^3 = 3xy$.

(i) Show that there is no point of inflection on C. You may assume that the origin is not a point of inflection.

(ii) The part of C which lies in the first quadrant is a closed loop touching the axes at the origin. By converting to polar coordinates, or otherwise, evaluate the area of this loop.

4 2008 S3 Q3

The point $P(a\cos\theta\,,\, b\sin\theta)$, where $a > b > 0$, lies on the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

The point S is located at $(-ea\,,\,0)$, where $b^2 = a^2(1-e^2)\,$.

The point N is the foot of the perpendicular from the origin, O, to the tangent to the ellipse at P. The lines SP and ON intersect at T. Show that the y-coordinate of T is

$$\frac{b\sin\theta}{1+e\cos\theta}\,.$$

Show that T lies on the circle with centre S and radius a.

This question has been edited very slightly to remove references to “ellipse” and “focus”.

5 2011 S3 Q5

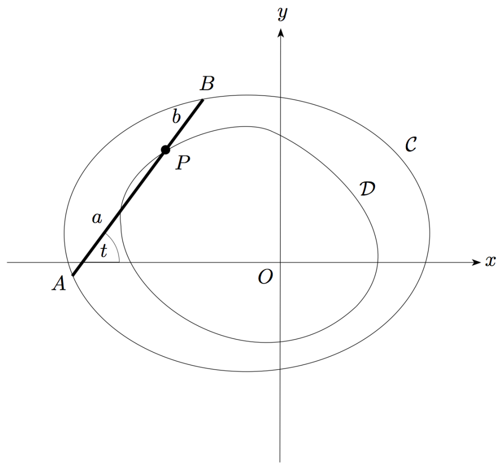
A movable point P has cartesian coordinates (x, y), where x and y are functions of t. The polar coordinates of P with respect to the origin O are r and θ. Starting with the expression

$$\tfrac{1}{2}\int r^2\,\mathrm{d}\theta$$

for the area swept out by OP, obtain the equivalent expression

$$\tfrac{1}{2}\int \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t} \right) \mathrm{d}t\,. \qquad (*)$$

The ends of a thin straight rod AB lie on a closed convex curve $\mathcal{C}$. The point P on the rod is a fixed distance a from A and a fixed distance b from B. The angle between AB and the positive x direction is t. As A and B move anticlockwise round $\mathcal{C}$, the angle t increases from 0 to 2π and P traces a closed convex curve $\mathcal{D}$ inside $\mathcal{C}$, with the origin O lying inside $\mathcal{D}$, as shown in the diagram.



Let (x, y) be the coordinates of P. Write down the coordinates of A and B in terms of a, b, x, y and t.

The areas swept out by OA, OB and OP are denoted by [A], [B] and [P], respectively. Show, using $(*)$, that

$$[A] = [P] + \pi a^2 - af$$

where

$$f = \tfrac{1}{2}\int_0^{2\pi} \left(\left(x + \frac{\mathrm{d}y}{\mathrm{d}t} \right)\cos t + \left(y - \frac{\mathrm{d}x}{\mathrm{d}t} \right)\sin t \right) \mathrm{d}t\,.$$

Obtain a corresponding expression for [B] involving b. Hence show that the area between the curves $\mathcal{C}$ and $\mathcal{D}$ is πab.