STEP Support Programme

STEP 3 Differential Equations: Hints

1 **(i)** Substituting the two given functions for u will give two simultaneous equations that you can solve for $\mathrm{a}(x)$ and $\mathrm{b}(x)$.

The equation therefore has a solution $u = x$ and a solution $u = \mathrm{e}^{-x}$ and so the general solution can be written as $u = \lambda x + \mu \mathrm{e}^{-x}$ [1].

When differentiating y remember to use the product rule, and remember that you are differentiating with respect to x, not u. The given substitution for y transforms the differential equation into the equation in u that is used at the start of the question, hence you can write down the general solution for u and use this to find y. The final step is to use the given condition to find the specific solution for y.

(ii) The differential equation here looks very similar to $(*)$. Start by using a substitution for y similar to the one used in part **(i)**. The differential equation in u will be different, so you will need to find a general solution for u in this case (but it will be very similar to the last part).

2 Don't forget to do the very first bit! For this, you need to differentiate y (twice), using the product and chain rules. One of the "required formulae" in the STEP specifications is $\dfrac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x = \frac{1}{\sqrt{(1-x^2)}}$.

For the next part, differentiate the given differential equation and gather terms. Then do it again! Looking at the three differential equations should lead you to a conjecture which can be proved using proof by induction.

The general form of Maclaurin's is another of the required formulae. You already have the first and second derivatives, so can find $\mathrm{f}(0)$, $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$. Unfortunately one of these is zero, so you need to do a bit more work, using the relationship between the derivatives proved earlier. Remember that you will be interested in the derivatives when $x = 0$.

For the series to be a polynomial it must terminate (i.e. not continue forever).

[1] Two specific solutions, y_1, y_2 of a second order homogeneous equation $y'' + \mathrm{p}(x)y' + \mathrm{q}(x) = 0$ can be used to define a *general* solution $y = \lambda y_1 + \mu y_2$ **if and only if** they are *linearly independent*, that is there are no possible non-zero coefficients c, k such that $cy_1(x) + ky_2(x) = 0$ for all x. Two solutions are linearly independent **iff** the Wronskian is not zero, i.e. $W(y_1 y_2)(x) = y_1 y_2' - y_1' y_2 \neq 0$. You don't need to prove linear independence in STEP, just make sure you don't use two obviously dependent functions such as $y = x$ and $y = 2x$.

3 Note that $\mathrm{Q}(x)\mathrm{R}'(x) - \mathrm{Q}'(x)\mathrm{R}(x)$ looks suspiciously like it might be related to the quotient rule.

(i) Start by finding a function $\mathrm{R}(x)$ which satisfies $\mathrm{P}(x) = \mathrm{Q}(x)\mathrm{R}'(x) - \mathrm{Q}'(x)\mathrm{R}(x)$ (using the $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ given in this part of the question). You should find that $\mathrm{R}(x)$ is not unique, but you can show that two different choices of $\mathrm{R}(x)$ give the same answer for $\displaystyle\int \frac{\mathrm{P}(x)}{(\mathrm{Q}(x))^2}\,\mathrm{d}x$.

(ii) Start by dividing throughout to get an equations of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{f}(x)y = \mathrm{g}(x)$. You can then find the associated integrating factor (which will look a bit messy at first, but should simplify fairly nicely). You should end up needing to find an integral of the form $\displaystyle\int \frac{\mathrm{P}(x)}{(\mathrm{Q}(x))^2}\,\mathrm{d}x$ again, so follow a similar method to before. Like in part **(i)**, $\mathrm{R}(x)$ will not be unique.

4 You can rearrange the first equation to get z in terms of y and $\dot{y}$, and then differentiate this with respect to t. You should be able to get a second order homogeneous differential equation with constant coefficients. Solve for y and then find z.

(i) Use the initial conditions to find two simultaneous equations in A and B, and hence find A and B.

(ii) Like part **(i)**. A and B will involve c and e. They will be quite messy-looking (but can be simplified somewhat).

(iii) Start by expressing $z_1(t-n)$ in terms of exponentials. e^{-t} can be taken "outside" the sum (as can some other bits). You will get some infinite sums that you can evaluate. Finally compare coefficients of e^{-t} and e^{-6t} to find the particular value of c.