STEP Support Programme

STEP 3 Differential Equations: Solutions

1 **(i)** Substituting $u = x$ gives:

$$0 + \mathrm{a}(x) + x\mathrm{b}(x) = 0\,.$$

Substituting $u = \mathrm{e}^{-x}$ gives:

$$\begin{aligned}\mathrm{e}^{-x} - \mathrm{a}(x)\mathrm{e}^{-x} + \mathrm{b}(x)\mathrm{e}^{-x} &= 0\\ 1 - \mathrm{a}(x) + \mathrm{b}(x) &= 0 \qquad \text{since } \mathrm{e}^{-x} \neq 0\,.\end{aligned}$$

Then substituting $\mathrm{a}(x) = 1 + \mathrm{b}(x)$ into the first equation gives:

$$\begin{aligned}1 + \mathrm{b}(x) + x\mathrm{b}(x) &= 0\\ \implies \mathrm{b}(x) &= \frac{-1}{1+x}\\ \implies \mathrm{a}(x) &= 1 + \frac{-1}{1+x}\\ &= \frac{x}{1+x}\end{aligned}$$

The general solution of the equation is $u = Ax + B\mathrm{e}^{-x}$.

Differentiating $y = \dfrac{1}{3u}\dfrac{\mathrm{d}u}{\mathrm{d}x}$ gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3u}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{1}{3u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2$$

Substituting into $(*)$:

$$\frac{1}{3u}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \cancel{\frac{1}{3u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2} + \cancel{3\left(\frac{1}{3u}\right)^2\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2} + \frac{x}{1+x} \times \frac{1}{3u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{3(1+x)}$$

Then multiplying throughout by $3u$ gives:

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{x}{1+x}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\cancel{3}u}{\cancel{3}(1+x)}$$

which is the required equation, which happens to be the first equation with the $\mathrm{a}(x)$ and $\mathrm{b}(x)$ you found earlier. This is probably not a coincidence.

The general solution of this is $u = Ax + B\mathrm{e}^{-x}$ (from before), so we have:

$$\begin{aligned}y &= \frac{1}{3u}\frac{\mathrm{d}u}{\mathrm{d}x}\\ &= \frac{1}{3} \times \frac{1}{Ax + B\mathrm{e}^{-x}} \times \left(A - B\mathrm{e}^{-x}\right)\\ &= \frac{A - B\mathrm{e}^{-x}}{3\left(Ax + B\mathrm{e}^{-x}\right)}\end{aligned}$$

Using the initial condition $y = 0$ at $x = 0$ gives:

$$0 = \frac{A - B}{3(B)}$$

so we have $A = B$ and $y = \dfrac{A - A\mathrm{e}^{-x}}{3\left(Ax + A\mathrm{e}^{-x}\right)} = \dfrac{1 - \mathrm{e}^{-x}}{3\left(x + \mathrm{e}^{-x}\right)}$.

(ii) A similar sort of substitution is likely to be a good idea here, and the differences between this equation and $(*)$ is the absence of "3"'s and a $1 - x$ appearing rather than $1 + x$.

Using a substitution of $y = \dfrac{1}{u}\dfrac{\mathrm{d}u}{\mathrm{d}x}$ gives:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + \frac{x}{1-x}y &= \frac{1}{1-x} \\
\frac{1}{u}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \cancel{\frac{1}{u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2} + \cancel{\left(\frac{1}{u}\right)^2\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2} + \frac{x}{1-x}\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{1}{1-x} \\
\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{x}{1-x}\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{u}{1-x} \\
\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{x}{1-x}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{u}{1-x} &= 0
\end{aligned}$$

This now looks very similar to the previous part. If you substitute in $u = x$ and $u = \mathrm{e}^x$ you will find that these both satisfy the equation in u, and so the general solution is $u = Cx + D\mathrm{e}^x$. This gives:

$$y = \frac{C + D\mathrm{e}^x}{Cx + D\mathrm{e}^x}$$

Using the initial condition $y = 2$ at $x = 0$ gives:

$$\begin{aligned}
2 &= \frac{C + D}{D} \\
2D &= C + D \\
D &= C
\end{aligned}$$

and so the solution is $y = \dfrac{1 + \mathrm{e}^x}{x + \mathrm{e}^x}$.

2 One of the "required formulae" is $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$ (See the STEP specifications for the complete list of required formulae). This means that you should either be able to recall this result or be able to derive it with little trouble (in this case write $y = x$ and use implicit differentiation).

If $y = \cos(m \arcsin x)$ we have $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\sin(m\arcsin x) \times \dfrac{m}{\sqrt{1-x^2}}$ and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\cos(m\arcsin x) \times \frac{m^2}{1-x^2} - m\sin(m\arcsin x) \times -\tfrac{1}{2} \times -2x \times (1-x^2)^{-3/2}$$

To simplify things, let $S = \sin(m\arcsin x)$ and $C = \cos(m\arcsin x)$. We then have:

$$\begin{aligned} y &= C \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{mS}{\sqrt{1-x^2}} \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\frac{m^2C}{1-x^2} - \frac{mxS}{(1-x^2)^{3/2}} \end{aligned}$$

Substituting these into the given equation we have:

$$\begin{aligned} \left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + m^2y &= \left(1-x^2\right)\left(-\frac{m^2C}{1-x^2} - \frac{mxS}{(1-x^2)^{3/2}}\right) - x\left(-\frac{mS}{\sqrt{1-x^2}}\right) + m^2C \\ &= -m^2C - \frac{mxS}{(1-x^2)^{1/2}} + \frac{xmS}{\sqrt{1-x^2}} + m^2C \\ &= 0 \end{aligned}$$

Hence $y = \cos(m\arcsin x)$ is a solution to the given differential equation.

Differentiating the given equation gives:

$$\begin{aligned} \left(1-x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} + m^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(1-x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(m^2-1\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

And then differentiating again gives:

$$\begin{aligned} \left(1-x^2\right)\frac{\mathrm{d}^4y}{\mathrm{d}x^4} - 2x\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 3x\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(m^2-1\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 0 \\ \left(1-x^2\right)\frac{\mathrm{d}^4y}{\mathrm{d}x^4} - 5x\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \left(m^2-4\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 0 \end{aligned}$$

Looking at the three differential equations (and also thinking about what happens as we differentiate) we make the hypothesis that:

$$\left(1-x^2\right)\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} - (2n+1)x\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} + (m^2-n^2)\frac{\mathrm{d}^ny}{\mathrm{d}x^n} = 0$$

Base case: we know that this is true when $n = 0$ (and also $n = 1, 2$).

Inductive step: assume that the hypothesis is true when $n = k$, i.e. we have:

$$\left(1-x^2\right)\frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} - (2k+1)x\,\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} + (m^2-k^2)\frac{\mathrm{d}^{k}y}{\mathrm{d}x^{k}} = 0 \qquad (*)$$

Differentiating $(*)$ with respect to x gives:

$$\begin{aligned}
\left(1-x^2\right)\frac{\mathrm{d}^{k+3}y}{\mathrm{d}x^{k+3}} - 2x\frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} - (2k+1)x\,\frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} - (2k+1)\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} + (m^2-k^2)\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} &= 0\\
\left(1-x^2\right)\frac{\mathrm{d}^{k+3}y}{\mathrm{d}x^{k+3}} - \left(2x+(2k+1)x\right)\frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} + \left(m^2-k^2-(2k+1)\right)\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} &= 0\\
\left(1-x^2\right)\frac{\mathrm{d}^{k+3}y}{\mathrm{d}x^{k+3}} - \left(2+2k+1\right)x\,\frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} + \left(m^2-(k^2+2k+1)\right)\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} &= 0\\
\left(1-x^2\right)\frac{\mathrm{d}^{k+3}y}{\mathrm{d}x^{k+3}} - \left(2(k+1)+1\right)x\,\frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} + \left(m^2-(k+1)^2\right)\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} &= 0
\end{aligned}$$

and this is the same as $(*)$ with $n = k+1$, Hence if it is true for $n = k$ then it is true for $n = k+1$, and as it is true for $n = 0$ it is true for all integers $n \geqslant 0$.

The general Maclaurin's Series expansion is:

$$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \ldots$$

Since $y = \cos(m \arcsin x)$, when $x = 0$ we have $y = 1$. Using the expressions found for the first two derivatives of y found at the start of the question, we have $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -m^2$ when $x = 0$.

Using $(1-x^2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (m^2-1)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ with $x = 0$ gives $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 0$.

Finally $(1-x^2)\frac{\mathrm{d}^4y}{\mathrm{d}x^4} - 5x\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (m^2-4)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ with $x = 0$ gives $\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = m^2(m^2-4)$.

Hence the first three non-zero terms of the Maclaurin series for y are:

$$y = 1 - \frac{m^2}{2!}x^2 + \frac{m^2(m^2-4)}{4!}x^4 + \ldots .$$

Letting $x = \sin\theta$ gives:

$$y = \cos(m\theta) = 1 - \frac{m^2}{2!}\sin^2\theta + \frac{m^2(m^2-4)}{4!}\sin^4\theta + \ldots$$

By considering

$$\left(1-x^2\right)\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} - (2n+1)x\,\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} + (m^2-n^2)\frac{\mathrm{d}^{n}y}{\mathrm{d}x^{n}} = 0$$

and letting $x = 0$, we have:

$$\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} = -(m^2-n^2)\frac{\mathrm{d}^{n}y}{\mathrm{d}x^{n}}$$

This means that all the odd differentials are going to be zero, and the even ones will be given by $(-1)^{k+1}m^2(m^2-2^2)(m^2-4^2)\ldots\left(m^2-(2k)^2\right)$. This means that if m is even this Maclaurin's series for $\cos m\theta$ will terminate since at some point all the derivatives will be zero. Hence the series is a polynomial in $\sin\theta$ and will have degree m (to convince yourself of this final bit, try considering $m = 2$, $m = 4$ etc.).

3 We have:

$$\begin{aligned}\int \frac{\mathrm{P}(x)}{\left(\mathrm{Q}(x)\right)^2}\,\mathrm{d}x &= \int \frac{\mathrm{Q}(x)\mathrm{R}'(x) - \mathrm{Q}'(x)\mathrm{R}(x)}{\left(\mathrm{Q}(x)\right)^2}\,\mathrm{d}x \\ &= \int \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{R}(x)}{\mathrm{Q}(x)}\right)\mathrm{d}x \\ &= \frac{\mathrm{R}(x)}{\mathrm{Q}(x)} + k\end{aligned}$$

(i) Let $\mathrm{P}(x) = 5x^2 - 4x - 3$ and let $\mathrm{Q}(x) = 1 + 2x + 3x^2$. We then want $\mathrm{R}(x) = a + bx + cx^2$ to satisfy:

$$\begin{aligned}\mathrm{P}(x) &= \mathrm{Q}(x)\mathrm{R}'(x) - \mathrm{Q}'(x)\mathrm{R}(x) \\ 5x^2 - 4x - 3 &= \left(1 + 2x + 3x^2\right)(b + 2cx) - (2 + 6x)\left(a + bx + cx^2\right)\end{aligned}$$

Note that the x^3 terms on the RHS cancel out. Equating coefficients gives:

$$\begin{aligned}-3 &= b - 2a \\ -4 &= 2b + 2c - 6a - 2b \\ 5 &= 3b + 4c - 2c - 6b\end{aligned}$$

Simplifying gives:

$$\begin{aligned}-3 &= b - 2a && (1) \\ -2 &= c - 3a && (2) \\ 5 &= 2c - 3b && (3)\end{aligned}$$

These three equations are not linearly independent, i.e. if we cancel a from the first two (by considering $2 \times (2) - 3 \times (1)$) we get:

$$(2 \times -2) - (3 \times -3) = 2c - 3b \implies 5 = 2c - 3b$$

Which is the same as (3) and hence the three solutions do not have a unique solution.

This means that we can pick a arbitrarily. Letting $a = 1$ gives $b = -1$ and $c = 1$ and hence the integral is equal to:

$$\frac{\mathrm{R}(x)}{\mathrm{Q}(x)} + k = \frac{1 - x + x^2}{1 + 2x + 3x^2} + k$$

If instead we took $a = 2$, we would have $b = 1$ and $c = 4$. This gives:

$$\frac{\mathrm{R}(x)}{\mathrm{Q}(x)} + k' = \frac{2 + x + 4x^2}{1 + 2x + 3x^2} + k'$$

We can re-write the second integral as follows:

$$\begin{aligned}\frac{2 + x + 4x^2}{1 + 2x + 3x^2} + k' &= \frac{(1 + 2x + 3x^2) + 1 - x + x^2}{1 + 2x + 3x^2} + k' \\ &= 1 + \frac{1 - x + x^2}{1 + 2x + 3x^2} + k' \\ &= \frac{1 - x + x^2}{1 + 2x + 3x^2} + k\end{aligned}$$

I.e. the only difference is in the arbitrary constant.

(ii) Start by rearranging the equation to get:

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\sin x-2\cos x}{1+\cos x+2\sin x}\,y=\frac{5-3\cos x+4\sin x}{1+\cos x+2\sin x}$$

The integrating factor is:

$$\begin{aligned}\mathrm{e}^{\int\frac{\sin x-2\cos x}{1+\cos x+2\sin x}\,\mathrm{d}x}&=\mathrm{e}^{-\ln(1+\cos x+2\sin x)}\\&=\frac{1}{1+\cos x+2\sin x}\end{aligned}$$

Multiplying by the integrating factor gives:

$$\begin{aligned}\frac{1}{1+\cos x+2\sin x}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\sin x-2\cos x}{(1+\cos x+2\sin x)^2}\,y&=\frac{5-3\cos x+4\sin x}{(1+\cos x+2\sin x)^2}\\\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1+\cos x+2\sin x}\times y\right)&=\frac{5-3\cos x+4\sin x}{(1+\cos x+2\sin x)^2}\\\frac{1}{1+\cos x+2\sin x}\times y&=\int\frac{5-3\cos x+4\sin x}{(1+\cos x+2\sin x)^2}\,\mathrm{d}x\end{aligned}$$

Now let $\mathrm{P}(x)=5-3\cos x+4\sin x$, $\mathrm{Q}(x)=1+\cos x+2\sin x$ and $\mathrm{R}(x)=a+b\cos x+c\sin x$. Using $\mathrm{P}(x)=\mathrm{Q}(x)\mathrm{R}'(x)-\mathrm{Q}'(x)\mathrm{R}(x)$ gives:

$$\begin{aligned}5-3\cos x+4\sin x&=(1+\cos x+2\sin x)\left(-b\sin x+c\cos x\right)\\&\qquad-\left(-\sin x+2\cos x\right)\left(a+b\cos x+c\sin x\right)\\&=\left(-b\sin x+c\cos x-2b\sin^2 x+c\cos^2 x+(2c-b)\sin x\cos x\right)\\&\quad-\left(-a\sin x+2a\cos x-c\sin^2 x+2b\cos^2 x+(2c-b)\sin x\cos x\right)\\&=(c-2b)\left(\sin^2 x+\cos^2 x\right)+(c-2a)\cos x+(a-b)\sin x\end{aligned}$$

So we have:

$$5=c-2b \qquad (4)$$
$$-3=c-2a \qquad (5)$$
$$4=a-b \qquad (6)$$

Note that (4) − (5) gives $8=-2b+2a \implies 4=a-b$. The equations are linearly dependent. Picking $a=5$, $b=1$ gives $c=7$[1]. Using $\displaystyle\int\frac{\mathrm{P}(x)}{\left(\mathrm{Q}(x)\right)^2}\,\mathrm{d}x=\frac{\mathrm{R}(x)}{\mathrm{Q}(x)}+k$ gives:

$$\begin{aligned}\int\frac{5-3\cos x+4\sin x}{(1+\cos x+2\sin x)^2}\,\mathrm{d}x&=\frac{5+\cos x+7\sin x}{1+\cos x+2\sin x}+k\\\implies\frac{y}{1+\cos x+2\sin x}&=\frac{5+\cos x+7\sin x}{1+\cos x+2\sin x}+k\\\implies y&=5+\cos x+7\sin x+(1+\cos x+2\sin x)k\end{aligned}$$

This answer is not unique, as we could rewrite it as

$$\begin{aligned}y&=1+5\sin x+(1+\cos x+2\sin x)+(1+\cos x+2\sin x)k\\&=1+5\sin x+(1+\cos x+2\sin x)(k+1)\\&=1+5\sin x+(1+\cos x+2\sin x)k'\end{aligned}$$

etc.

[1] Note that you could have picked any value of a you liked.

4 There are various ways you can find the differential equation in y. The following is one possible approach.

Firstly, we have $\dot{y} = -2y + 2z$. This gives:

$$\begin{aligned} z &= \frac{\dot{y}}{2} + y \\ \implies \dot{z} &= \frac{\ddot{y}}{2} + \dot{y} \end{aligned}$$

and so

$$\begin{aligned} \dot{z} &= -\dot{y} - 3z \\ \implies \frac{\ddot{y}}{2} + \dot{y} &= -\dot{y} - 3\left(\frac{\dot{y}}{2} + y\right) \\ \ddot{y} + 2\dot{y} &= -2\dot{y} - 3\left(\dot{y} + 2y\right) \\ \ddot{y} + 7\dot{y} + 6y &= 0 \end{aligned}$$

This gives the characteristic polynomial/ auxiliary equation $r^2 + 7r + 6 = 0$ which has roots $r = -1, -6$. The general equation is therefore

$$\begin{aligned} y &= A\mathrm{e}^{-t} + B\mathrm{e}^{-6t} \implies \\ z &= \frac{\dot{y}}{2} + y \\ &= -\tfrac{1}{2}A\mathrm{e}^{-t} - 3B\mathrm{e}^{-6t} + A\mathrm{e}^{-t} + B\mathrm{e}^{-6t} \\ &= \tfrac{1}{2}A\mathrm{e}^{-t} - 2B\mathrm{e}^{-6t} \end{aligned}$$

(i) Using $z(0) = 0$ and $y(0) = 5$ gives us:

$$0 = \tfrac{1}{2}A - 2B \qquad \text{and} \qquad 5 = A + B$$

Solving these simultaneously gives $A = 4$ and $B = 1$, and so:

$$y_1(t) = 4\mathrm{e}^{-t} + \mathrm{e}^{-6t} \qquad \text{and} \qquad z_1(t) = 2\mathrm{e}^{-t} - 2\mathrm{e}^{-6t}$$

(ii) $z(0) = z(1) = c$ gives:

$$\tfrac{1}{2}A - 2B = c \qquad \text{and} \qquad \tfrac{1}{2}A\mathrm{e}^{-1} - 2B\mathrm{e}^{-6} = c$$

Hence:

$$\tfrac{1}{2}A - 2B = c \tag{7}$$

$$\tfrac{1}{2}A - 2B\mathrm{e}^{-5} = c \times \mathrm{e} \tag{8}$$

$$(2) - (1) \implies 2B\left(1 - \mathrm{e}^{-5}\right) = c(\mathrm{e} - 1) \tag{9}$$

and so

$$B = \frac{c(\mathrm{e} - 1)}{2\left(1 - \mathrm{e}^{-5}\right)} = \frac{c\left(\mathrm{e}^6 - \mathrm{e}^5\right)}{2\left(\mathrm{e}^5 - 1\right)}$$

and

$$\begin{aligned}
A &= 2c + 4B \\
&= 2c + \frac{2c\left(e^6 - e^5\right)}{\left(e^5 - 1\right)} \\
&= 2c\left(1 + \frac{\left(e^6 - e^5\right)}{\left(e^5 - 1\right)}\right) \\
&= 2c\left(\frac{e^5 - 1 + e^6 - e^5}{e^5 - 1}\right) \\
&= 2c\left(\frac{e^6 - 1}{e^5 - 1}\right)
\end{aligned}$$

We therefore have:

$$\begin{aligned}
y_2(t) &= 2c\left(\frac{e^6 - 1}{e^5 - 1}\right)e^{-t} + \frac{c\left(e^6 - e^5\right)}{2\left(e^5 - 1\right)}e^{-6t} \qquad \text{and} \\
z_2(t) &= c\left(\frac{e^6 - 1}{e^5 - 1}\right)e^{-t} - \frac{c\left(e^6 - e^5\right)}{\left(e^5 - 1\right)}e^{-6t}
\end{aligned}$$

(iii) We have $z_1(t-n) = 2e^{-(t-n)} - 2e^{-6(t-n)} = 2e^{-t}e^{n} - 2e^{-6t}e^{6n}$. This means we have:

$$\begin{aligned}
\sum_{n=-\infty}^{0} z_1(t-n) &= 2e^{-t}\sum_{n=-\infty}^{0} e^{n} - 2e^{-6t}\sum_{n=-\infty}^{0} e^{6n} \\
&= 2e^{-t}\sum_{i=0}^{\infty} e^{-i} - 2e^{-6t}\sum_{i=0}^{\infty} e^{-6i}
\end{aligned}$$

The last step is taken by letting the index $n = -i$ (and reversing the sum). The first sum is the infinite sum of a Geometric Progression with common ratio e^{-1} and the second is a GP with common ratio e^{-6}. We therefore have:

$$\begin{aligned}
\sum_{n=-\infty}^{0} z_1(t-n) &= 2e^{-t} \times \frac{1}{1 - e^{-1}} - 2e^{-6t} \times \frac{1}{1 - e^{-6}} \\
&= \frac{2e}{e - 1}e^{-t} - \frac{2e^6}{e^6 - 1}e^{-6t} \qquad (\dagger)
\end{aligned}$$

Comparing the coefficient of e^{-t} of this to the coefficient of e^{-t} from $z_2(t)$ we need:

$$\begin{aligned}
c\left(\frac{e^6 - 1}{e^5 - 1}\right) &= \frac{2e}{e - 1} \\
\implies c &= \frac{2e}{e - 1} \times \frac{\left(e^5 - 1\right)}{\left(e^6 - 1\right)}
\end{aligned}$$

This means that the coefficient of e^{-6t} in $z_2(t)$ is:

$$\begin{aligned}
-\frac{c\left(\mathrm{e}^6-\mathrm{e}^5\right)}{\left(\mathrm{e}^5-1\right)} &= -\frac{2\mathrm{e}}{\mathrm{e}-1}\times\frac{\left(\mathrm{e}^5-1\right)}{\left(\mathrm{e}^6-1\right)}\times\frac{\left(\mathrm{e}^6-\mathrm{e}^5\right)}{\left(\mathrm{e}^5-1\right)} \\
&= -\frac{2\mathrm{e}}{\cancel{(\mathrm{e}-1)}}\times\frac{\cancel{(\mathrm{e}^5-1)}}{\left(\mathrm{e}^6-1\right)}\times\frac{\mathrm{e}^5\cancel{(\mathrm{e}-1)}}{\cancel{(\mathrm{e}^5-1)}} \\
&= -\frac{2\mathrm{e}^6}{\mathrm{e}^6-1}
\end{aligned}$$

which is the same as the coefficient of e^{-6t} in (†) and hence we have

$$z_2(t) = \sum_{n=-\infty}^{0} z_1(t-n)$$

when

$$c = \frac{2\mathrm{e}}{\mathrm{e}-1}\times\frac{\left(\mathrm{e}^5-1\right)}{\left(\mathrm{e}^6-1\right)}\,.$$