

## STEP Support Programme

### STEP 3 Differential Equations Topic Notes

#### First Order Linear Differential Equations

A first order differential equation has  $\frac{dy}{dx}$  as the highest derivative. It is linear if the  $\frac{dy}{dx}$  and  $y$  terms (where they appear) have order 1. The equation will be *homogeneous* if there are no terms that do not contain  $y$ .

A general First Order Linear DE can be written as:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

and if  $Q(x) \equiv 0$  then this will be a homogeneous equation. Note that  $P(x)$  and  $Q(x)$  can be non-linear, i.e. the equation is *linear in y*.

There are various methods you can use to solve these:

- **“Simple” integration** If you have an equation of the form  $\frac{dy}{dx} = Q(x)$  then the solution is  $y = \int Q(x)dx$ .
- **Separation of variables** If the equation can be written as  $\frac{dy}{dx} = f(x)g(y)$  then it is said to be *separable*. (See STEP 2 Calculus module).
- **Integrating Factors** If you have an equation of the form  $\frac{dy}{dx} + P(x)y = Q(x)$  then you can multiply throughout by the integrating factor  $e^{\int P(x)dx}$ . This then gives:

$$\begin{aligned} e^{\int P(x)dx} \times \frac{dy}{dx} + P(x)e^{\int P(x)dx} \times y &= e^{\int P(x)dx} \times Q(x) \\ \frac{d}{dx} \left( e^{\int P(x)dx} \times y \right) &= e^{\int P(x)dx} \times Q(x) \\ e^{\int P(x)dx} \times y &= \int \left( e^{\int P(x)dx} \times Q(x) \right) dx \end{aligned}$$

#### Second Order Differential Equations

The general second order differential equation with constant coefficients looks like:

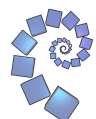
$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \quad (*)$$

and if  $f(x) \equiv 0$  then this is said to be a *homogeneous* equation (otherwise it is *in-homogeneous*). The solution to a homogeneous differential equation is also called a *Complementary Function*.

The **auxiliary equation** can be used to solve the *homogeneous* equation  $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$ . It is found by substituting  $y = Ke^{\lambda x}$  into the equation which gives:

$$\begin{aligned} aK\lambda^2 e^{\lambda x} + bK\lambda e^{\lambda x} + cKe^{\lambda x} &= 0 \quad \text{i.e.} \\ a\lambda^2 + b\lambda + c &= 0 \end{aligned}$$

This quadratic equation can then be solved to find the possible values of  $\lambda$ .



- If the auxiliary equation has two real roots then the solution is  $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$ .
- If the auxiliary equation has a repeated root  $\lambda_1$  then the solution is  $y = (A + Bx)e^{\lambda_1 x}$ .
- If the auxiliary equation has the complex roots  $\lambda = \alpha \pm i\beta$  then the solution is  $y = (A \sin \beta x + B \cos \beta x) e^{\alpha x}$ .

The general solution of the non-homogeneous second order linear differential equation (\*) with constant coefficients is the sum of the **complementary function** (i.e. the solution of the homogeneous version of the equation) and a **particular integral**.

To find the **particular integral**, use a trial function as follows:

- If  $f(x)$  is a polynomial of degree  $n$ , try  $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ .
- If  $f(x)$  involves  $e^{px}$ , try  $y = ce^{px}$ .
- If  $f(x)$  involves  $\sin px$  and  $\cos px$ , try  $y = \gamma \sin px + \delta \cos px$ .

### Substitutions

A lot of differential equations cannot be solved by these methods **BUT** if a clever substitution is used they can often be turned into a differential equation that **can** be solved. Often you are given these, or given the first one and then have to work out which substitution to use for following equations.

### Coupled Differential Equations

Sometimes you might be given two differential equations involving one independent variable (such as time) and two dependent variables. One example might be:

$$\begin{aligned}\frac{dx}{dt} &= 4x - 2y \\ \frac{dy}{dt} &= 3x - y\end{aligned}$$

In this case the two equations can be combined to make a second order differential equation in one dependent variable (start by differentiating one of them).

One type of coupled differential equations are “**Predator-Prey systems**”. Often *non-linear* differential equations are used to model these, such as the **Lotka-Volterra equations**.

### Simple Harmonic Motion

There are notes on Simple Harmonic motion in the STEP 3 Mechanics module. Note that you might be asked to solve an equation of the form  $\ddot{x} = -\alpha^2 x$  in a pure question (and not be told that it is the standard SHM question).

