STEP Support Programme

STEP 3 Differential Equations Topic Notes

First Order Linear Differential Equations

A first order differential equation has $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as the highest derivative. It is linear if the $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and y terms (where they appear) have order 1. The equation will be *homogeneous* if there are no terms that do not contain y.

A general First Order Linear DE can be written as:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{P}(x)y = \mathrm{Q}(x)$$

and if $\mathrm{Q}(x) \equiv 0$ then this will be a homogeneous equation. Note that $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ can be non-linear, i.e. the equation is *linear in y*.

There are various methods you can use to solve these:

- **"Simple" integration** If you have an equation of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{Q}(x)$ then the solution is $y = \int \mathrm{Q}(x)\mathrm{d}x$.
- **Separation of variables** If the equation can be written as $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)\mathrm{g}(y)$ then it is said to be *separable*. (See STEP 2 Calculus module).
- **Integrating Factors** If you have an equation of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{P}(x)y = \mathrm{Q}(x)$ then you can multiply throughout by the integrating factor $\mathrm{e}^{\int \mathrm{P}(x)\mathrm{d}x}$. This then gives:

$$\begin{aligned} \mathrm{e}^{\int \mathrm{P}(x)\mathrm{d}x} \times \frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{P}(x)\mathrm{e}^{\int \mathrm{P}(x)\mathrm{d}x} \times y &= \mathrm{e}^{\int \mathrm{P}(x)\mathrm{d}x} \times \mathrm{Q}(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\int \mathrm{P}(x)\mathrm{d}x} \times y\right) &= \mathrm{e}^{\int \mathrm{P}(x)\mathrm{d}x} \times \mathrm{Q}(x) \\ \mathrm{e}^{\int \mathrm{P}(x)\mathrm{d}x} \times y &= \int \left(\mathrm{e}^{\int \mathrm{P}(x)\mathrm{d}x} \times \mathrm{Q}(x)\right)\mathrm{d}x \end{aligned}$$

Second Order Differential Equations

The general second order differential equation with constant coefficients looks like:

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = \mathrm{f}(x) \qquad (*)$$

and if $\mathrm{f}(x) \equiv 0$ then this is said to be a *homogeneous* equation (otherwise it is *in-homogeneous*). The solution to a homogeneous differential equation is also called a *Complementary Function*.

The **auxiliary equation** can be used to solve the *homogeneous* equation $a\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\dfrac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$. It is found by substituting $y = K\mathrm{e}^{\lambda x}$ into the equation which gives:

$$aK\lambda^2\mathrm{e}^{\lambda x} + bK\lambda\mathrm{e}^{\lambda x} + cK\mathrm{e}^{\lambda x} = 0 \qquad \text{i.e.}$$
$$a\lambda^2 + b\lambda + c = 0$$

This quadratic equation can then be solved to find the possible values of λ.

- If the auxiliary equation has two real roots then the solution is $y = A\mathrm{e}^{\lambda_1 x} + B\mathrm{e}^{\lambda_2 x}$.
- If the auxiliary equation has a repeated root λ_1 then the solution is $y = (A + Bx)\mathrm{e}^{\lambda_1 x}$.
- If the auxiliary equation has the complex roots $\lambda = \alpha \pm i\beta$ then the solution is $y = (A \sin \beta x + B \cos \beta x)\,\mathrm{e}^{\alpha x}$.

The general solution of the non-homogeneous second order linear differential equation $(*)$ with constant coefficients is the sum of the **complementary function** (i.e. the solution of the homogeneous version of the equation) and a **particular integral**.

To find the **particular integral**, use a trial function as follows:

- If $\mathrm{f}(x)$ is a polynomial of degree n, try $y = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$.
- If $\mathrm{f}(x)$ involves e^{px}, try $y = c\mathrm{e}^{px}$.
- If $\mathrm{f}(x)$ involves $\sin px$ and $\cos px$, try $y = \gamma \sin px + \delta \cos px$.

Substitutions

A lot of differential equations cannot be solved by these methods **BUT** if a clever substitution is used they can often be turned into a differential equation that **can** be solved. Often you are given these, or given the first one and then have to work out which substitution to use for following equations.

Coupled Differential Equations

Sometimes you might be given two differential equations involving one independent variable (such as time) and two dependent variables. One example might be:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x - 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3x - y$$

In this case the two equations can be combined to make a second order differential equation in one dependent variable (start by differentiating one of them).

One type of coupled differential equations are "Predator-Prey systems". Often *non-linear* differential equations are used to model these, such as the Lotka-Volterra equations.

Simple Harmonic Motion

There are notes on Simple Harmonic motion in the STEP 3 Mechanics module. Note that you might be asked to solve an equation of the form $\ddot{x} = -\alpha^2 x$ in a pure question (and not be told that it is the standard SHM question).