STEP Support Programme

STEP 2 Equations Questions: Hints

1  This question does not ask you to sketch any graphs, but it might be a very good idea to!

   (i) You could sketch the curve \( y = (x - 1)^4 + (x + 1)^4 \) and then consider how many times it would intersect \( y = c \) for different values of \( c \). You will have to carefully justify the number of turning points. Perhaps a clearer method is to consider a quadratic equation.

   (ii) Again, a sketch could work. Or you might be able to relate this to part (i).

   (iii) Definitely a sketch here! Your graph should consist of three straight lines. Remember that:

\[
|x - a| = \begin{cases} 
  x - a & \text{for } x > a \\
  a - x & \text{for } x < a.
\end{cases}
\]

   (iv) Another sketch. Remember that cubics must have at least one root.

2  For the first part, \( e = e^1 \). Show that \( 4! > 2^4 \) and then consider what each side will be multiplied by to get \( 5! \) and \( 2^5 \) etc. (you can use proof by induction if you wish).

You don’t need to find the coordinates of the minimum, you can look at the sign of the gradient for \( x = \frac{1}{2} \) and \( x = 1 \). You can also think about what happens to the gradient as \( x \to \frac{2}{3} \) (from below, as the curve is undefined for \( x \geq \frac{2}{3} \)).
3 (i) This is an “if and only if” so be careful! Start by trying to solve for $y$ and $z$ using the first two equations, and then if all three have a solution then $b = \cdots$. Then go the other way — i.e. show that if $b = 11$ then the equations have a solution.

(ii) If you try to eliminate a variable (probably $y$ or $z$) you will find that you end up with the same equation twice. Instead, try setting $z = \lambda$ and then use two equations to find $x$ and $y$ in terms of $a$ and $\lambda$. You must check the third equation!

(iii) Use your previous answer with $a = 2$ to write $x^2 + y^2 + z^2$ in terms of $\lambda$. You can then minimise this (Calculus is perhaps not the most efficient method here).

(iv) Use the general solution from part (ii). The condition $y^2 + z^2 < 1$ will give a condition on $\lambda$.

4 (i) Find the coordinates of the turning points. Then show whether they are above or below the $x$-axis. Remember that something squared is always greater than or equal to zero.

(ii) You can use the quadratic formula to find the two possible values of $u^3$. These both give the same value of $x$ (which is to be expected as there is only one real root of $x$).

(iii) Use $t^2 - pt + q \equiv (t - \alpha)(t - \beta)$ and it will be helpful to consider $(\alpha + \beta)^3$. The condition on the roots means that either $\alpha^2 = \beta$ or $\beta^2 = \alpha$. There is a connection back to the first two parts of the question.