# STEP Support Programme

## STEP 2 Equations Questions: Solutions

These are not fully worked solutions — you need to fill in the gaps. The questions do not ask you to sketch any graphs, but it might be a very good idea to!

**1** **(i)** First thing to notice is that $(x-1)^4+(x+1)^4>0$ (it cannot equal zero as then both brackets would have to be zero at the same time).

If you expand and simplify you get the equation $2x^4+12x^2+2-c=0$. Letting $x^2=t$ gives $2t^2+12t+2-c=0$, and we must have $t \geqslant 0$ (as $x$ is real and $x^2=t$). Solving for $t$ gives $t=\frac{-12\pm\sqrt{144-8(2-c)}}{4}$.

The negative square root will always give a negative value of $t$, so we need $-12+\sqrt{144-8(2-c)} \geqslant 0$, or equivalently $144-8(2-c) \geqslant 144$, which means that we need $c \geqslant 2$. When $c=2$, $t=x^2=0$ so we have just one solution, $x=0$. Also, there are two solutions if $c>2$ ($x^2=t$ means that $x=\pm\sqrt{t}$) and none if $c<2$.

Otherwise you can sketch the graph of $y=(x-1)^4+(x+1)^4$. If you differentiate you find that $\frac{\mathrm{d}y}{\mathrm{d}x}=x^3+3x=x(x^2+3)$, so there is only one turning point at $(0,2)$. Furthermore $x^2+3>0$ for all $x$ so we have $\frac{\mathrm{d}y}{\mathrm{d}x}<0$ when $x<0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}>0$ when $x>0$, so this one turning point is a minimum (or you can look at the sign of the second derivative). You can then sketch the graph and show that it intersects the line $y=c$ twice if $c>2$, once if $c=2$ (as then it passes through the vertex) and not at all if $c<2$.

**(ii)** This graph is simply a translation of the previous one by 2 units to the right — if $(x-1)^4+(x+1)^4=\mathrm{f}(x)$ then $(x-3)^4+(x-1)^4=\mathrm{f}(x-2)$. The answer will therefore be the same the one given in part **(i)**.

**(iii)** When drawing $y=|x-3|+|x-1|$ there are "critical values" at $x=1$ and $x=3$. This means that:

$$y=\begin{cases}-(x-3)-(x-1)=4-2x & \text{for } x<1\\ -(x-3)+(x-1)=2 & \text{for } 1\leqslant x\leqslant 3\\ (x-3)+(x-1)=2x+4 & \text{for } x>3\end{cases}$$

So there are no solutions for $c<2$, two solutions for $c>2$ and infinitely many solutions when $c=2$ (as then all the values of $x$ such that $1\leqslant x\leqslant 3$ satisfy the equation).

**(iv)** To sketch $y=(x-3)^3+(x-1)^3$, first find the derivative. This gives $\frac{\mathrm{d}y}{\mathrm{d}x}=6x^2-24x+30$ which can be written as $6(x-2)^2+6$. Hence the gradient is always positive, and is least when $x=2$. No turning points so exactly one root for any value of $c$.

For parts **(iii)** and **(iv)** you could treat them as a translation of a function in $(x+1)$ and $(x-1)$, but that doesn't seem to make the question much easier!

**2** $\mathrm{e}^1 = 1 + 1 + \dfrac{1^2}{2!} + \dfrac{1^3}{3!} + \dfrac{1^4}{4!} + \cdots$ which means $\mathrm{e} > 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} = \dfrac{8}{3}$.

When $n = 4$ we have $4! = 24$ and $2^4 = 16$ so we have $n! > 2^n$. One argument would then be now consider $(m+4)!$ and $2^{m+4}$ where $m$ is an integer with $m \geqslant 1$. The first one is the product of $4!$ and $m$ integers, each of which is greater than 4 and the second is the product of $2^4$ and $m$ more factors of 2. Hence $(m+4)! > 2^{m+4}$.

Alternative we can use a formal induction argument. We have already shown that $n! > 2^n$ when $n = 4$, now assume that is it true when $n = k$ i.e. we have $k! > 2^k$ (and $k \geqslant 4$). Now we have:

$$(k+1)! = (k+1) \times k! > 2 \times k! > 2 \times 2^k = 2^{k+1} .$$

(The first inequality sign is because if $k \geqslant 4$ then we have $k + 1 > 2$). Hence if it is true for $n = k$, it is true for $n = k + 1$ and as it is true for $n = 4$ it is true for all integers $n \geqslant 4$.

This means that for $n \geqslant 4$ we have $\frac{1}{n!} < \frac{1}{2^n}$. So we have:

$$\mathrm{e} < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots .$$

The first 4 terms sum to $\frac{8}{3}$. The rest form a geometric series which has a sum of $\frac{1}{8}$, so we have $\mathrm{e} < \frac{8}{3} + \frac{1}{8} = \frac{67}{24}$.

Differentiation gives $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6\mathrm{e}^{2x} - 14 \times \dfrac{1}{\frac{4}{3} - x}$. Trying to find the coordinates of the stationary point(s) is not easy, but what we can do is look at the sign of the gradient for $x = \frac{1}{2}$ and $x = 1$.

When $x = \frac{1}{2}$ we have:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\mathrm{e} - 14 \times \frac{6}{5} < 6 \times \frac{67}{24} - \frac{84}{5} = -\frac{1}{20} < 0$$

and hence the gradient is negative for $x = \frac{1}{2}$.

When $x = 1$ we have:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\mathrm{e}^2 - 14 \times 3 > 6 \times \left(\frac{8}{3}\right)^2 - 42 = \frac{2}{3} > 0$$

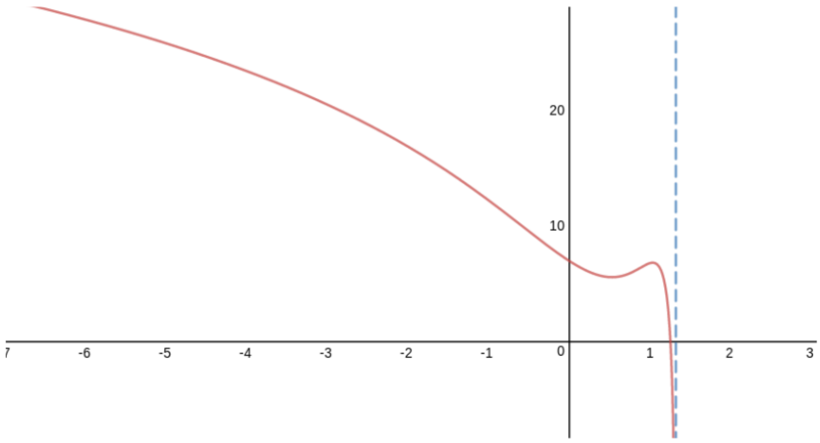
and the gradient is positive for $x = 1$, therefore there is a minimum between $x = \frac{1}{2}$ and $x = 1$.

As $x \to -\infty$, we have $y \approx 14\ln(-x)$. The graph is undefined for $x \geqslant \frac{4}{3}$ and as $x \to \frac{4}{3}$ we have $y \to -\infty$. This, along with the minimum between $x = \frac{1}{2}$ and $x = 1$ suggest that there will be a maximum between $x = 1$ and $x = \frac{4}{3}$, and you can show that when $x = \frac{5}{4}$ we have:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\mathrm{e}^{\frac{5}{2}} - 14 \times 12 < 6 \times 27 - 14 \times 12 = -6$$

using $\mathrm{e} < 3$ and $\mathrm{e}^{\frac{5}{2}} < \mathrm{e}^3$. Therefore the gradient is negative when $x = \frac{5}{4}$ and there is a maximum point between $x = 1$ and $x = \frac{5}{4}$.

The graph looks like this:



**3** **(i)** The question is an "if and only if" so you need to show that if the equations have a solution then $b = 11$ (the "only if" part) and if $b = 11$ then the equations have a solution.

Start with the "only if" by setting $a = 0$ and solving the first two equations. This gives $y = -1$ and $z = -2$ (and $x$ can be anything). If this is to be a solution of the set of three equations then the third one must be satisfied as well, so we need $-(-1) - 5 \times (-2) = b$ which gives $b = 11$.

Then if $b = 11$ we can show that the three equations have a solution (by substituting in $x = 1$, $y = -1$, $z = -2$ and showing that these values satisfy all three equations).

Actually the equations have infinitely many solutions when $b = 11$, "a solution" does not mean "exactly one".

**(ii)** Let $z = \lambda$. You can then use the first two equations to show that $x = \frac{4+2\lambda}{a}$ and $y = 1 + \lambda$. You then need to check that these satisfy the third equation.

**(iii)** If $a = 2$ and $b = 11$ then anything of the form $x = 2 + \lambda$, $y = 1 + \lambda$, $z = \lambda$ will be a solution. This gives:

$$x^2 + y^2 + z^2 = (2 + \lambda)^2 + (1 + \lambda)^2 + \lambda^2 = 3\lambda^2 + 6\lambda + 5\,.$$

To minimise this you can differentiate with respect to $\lambda$ and set the derivative equal to 0, but then you do need to show that this value of $\lambda$ gives a minimum (instead of a maximum, for example). It is perhaps simpler to write $3\lambda^2 + 6\lambda + 5 = 3(\lambda + 1)^2 + 2$ and then you can see that this will be minimised when $\lambda = -1$.

Hence we have $x = 1$, $y = 0$ and $z = -1$.

**(iv)** Let $b = 11$ again so that we know a solution exists, and we know we can write it as in part **(ii)**. Then we have $y^2 + z^2 = (1 + \lambda)^2 + \lambda^2$ and the condition $y^2 + z^2 < 1$ means that $\lambda^2 + \lambda < 0$ and hence we need $-1 < \lambda < 0$. So we could take $\lambda = -\frac{1}{2}$, which will give $y = \frac{1}{2}$, $z = -\frac{1}{2}$ and $x = \frac{3}{a}$. A possible value of $a$ is $10^{-6}$.

**4** **(i)** Differentiation gives $\frac{dy}{dx} = 3x^2 - 3q$ so the stationary points satisfy $x^2 = q$ and are at $(\sqrt{q},\, -2q\sqrt{q} - q(1+q))$ and $(-\sqrt{q},\, 2q\sqrt{q} - q(1+q))$. The $y$ coordinate of the first of these is obviously negative if $q > 0$ (but this still should be stated!). The $y$ coordinate of the second one of these can be written as $-q(1 + q - 2\sqrt{q}) = -q\left(1 - \sqrt{q}\right)^2$ and so this is also negative (since we are told that $q \neq 1$, otherwise this point would be on the $x$ axis and there would be two points of intersection of the curve with the $x$ axis). Hence both turning points of the cubic lie below the $x$ axis and the curve only crosses the $x$ axis once.

**(ii)** Substituting $x = u + q/u$ into the equation for $x$ and simplifying gives the equation in $u$ as $(u^3)^2 - q(1+q)u^3 + q^3 = 0$. Solving for $u^3$ gives:

$$u^3 = \frac{q(1+q) \pm \sqrt{q^2(1+q)^2 - 4q^3}}{2}\,.$$

The part in the square brackets is equal to $q^4 - 2q^3 + q^2 = q^2(1-q)^2$, and so we have $u^3 = q$ or $q^2$ i.e. $u = q^{\frac{1}{3}}$ or $q^{\frac{2}{3}}$. Both of these values of $u$ give the same value of $x$, which is good as we have shown that there is only one possible value in part **(i)**, i.e. $x = q^{\frac{1}{3}} + q^{\frac{2}{3}}$.

**(iii)** Since $t^2 - pt + q \equiv (t-\alpha)(t-\beta)$ we have $\alpha\beta = q$ and $\alpha + \beta = p$.

We also have $(\alpha+\beta)^3 = \alpha^3 + \beta^3 + 3\alpha^2\beta + 3\alpha\beta^2 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta)$.
This means that $p^3 = \alpha^3 + \beta^3 + 3qp$.

Since one of the roots is the square of the other we know that either $\alpha^2 = \beta$ or $\beta^2 = \alpha$. Hence we have $(\alpha^2 - \beta)(\beta^2 - \alpha) = 0$. Hence $\alpha^2\beta^2 + \alpha\beta - \alpha^3 - \beta^3 = 0$ and so $q^2 + q - (p^3 - 3qp) = 0$. This can be written as $p^3 - 3qp - q(1+q) = 0$ which looks suspiciously like something that appears in parts **(i)** and **(ii)**, just with $p$ instead of $x$. Then with the given conditions on $q$ we have the same situation as in part **(ii)** and so $p = q^{\frac{1}{3}} + q^{\frac{2}{3}}$.