STEP Support Programme

STEP 2 Equations and Inequalities Topic Notes

Equations

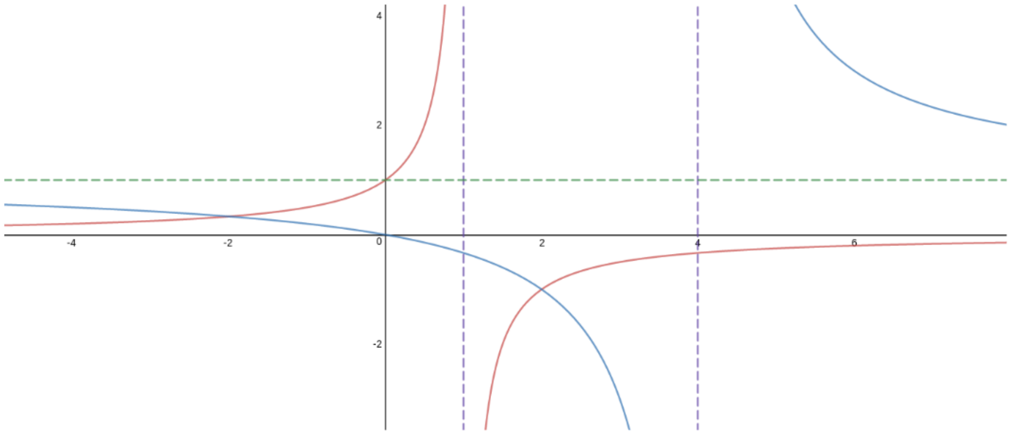
- For a quadratic equation, use the discriminant $\left(b^2 - 4ac\right)$ to find how many real roots the equation has.
- Substitutions can be used to turn a complicated equation into a simpler one (usually given to you, $x = t + \frac{1}{t}$ is a common one). Don't forget to find the values of the original variable.
- For other equations you could **sketch a graph** to work out how many real roots there are. For example, to find out how many real roots of the equation $x^3 - 3x + 1 = 0$ there are you could sketch $y = x^3 - 3x + 1$ (locate the turning points!) and see how many times it intersects the x axis. You could instead draw two graphs (e.g. $y = x^3 - 3x$ and $y = -1$) and see how many times they intersect.
- If $a < b$ and a *continuous* function $\mathrm{f}(x)$ has $\mathrm{f}(a) < 0$ and $\mathrm{f}(b) > 0$, (or vice versa) then there is a root of the equation in the interval $[a, b]$. This is a special case of the intermediate value theorem.
- If a cubic has roots α, β and γ then we can write:
$$x^3 + bx^2 + cx + d \equiv (x - \alpha)(x - \beta)(x - \gamma)\,.$$
Expanding the brackets and equating coefficients gives us:
$$\begin{aligned} \alpha + \beta + \gamma &= -b \\ \alpha\beta + \beta\gamma + \gamma\alpha &= c \\ \alpha\beta\gamma &= -d\,. \end{aligned}$$
You then then work out things like $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$
Expand $(\alpha + \beta + \gamma)^2$ to see why this is true.
There are similar equations for other polynomials.

Inequalities

- You can sometimes use completing the square to show that something is always positive or negative.
- If trying to show that $A \geqslant B$ it might be easier to show that $A - B \geqslant 0$. For example to show that $p^2 + q^2 \geqslant 2pq$ try showing that $p^2 + q^2 - 2pq \geqslant 0$.
- If $A > B$ and $C > D$ then $A + C > B + D$.
- If $A > B$ then $\dfrac{1}{A} < \dfrac{1}{B}$.
- Do **not** multiply or divide an inequality by something involving x (or another variable). You might be multiplying by something negative! (Addition or subtraction is fine).
- When solving inequalities, **graphs are your friends!** Replace the inequality with an equals sign, solve to find the intersections and then use your graph to solve the inequality.

Example 1: $\dfrac{1}{1-x} < \dfrac{x}{x-4}$

Start by sketching the two graphs $y = \dfrac{1}{1-x}$ and $y = \dfrac{x}{x-4}$.

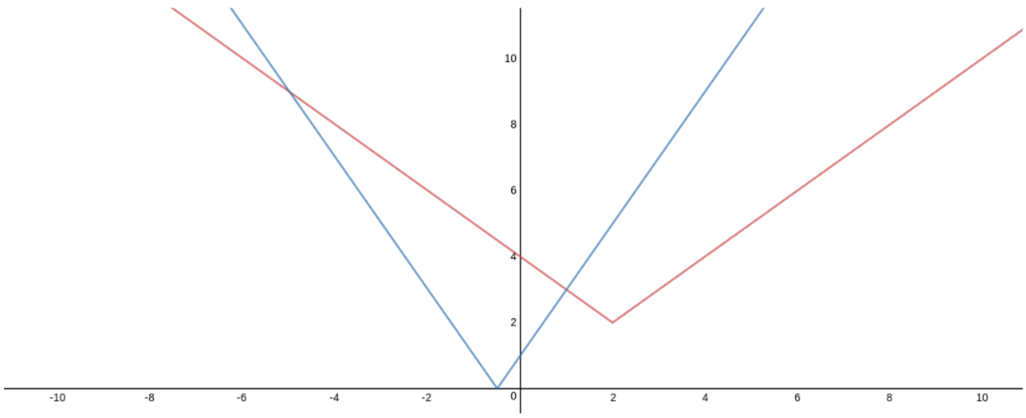


Solving the equation $\dfrac{1}{1-x} = \dfrac{x}{x-4}$ gives the points of intersection as $x = \pm 2$. You can then use the graph to see where the blue line is above the red line to give a final answer of $x < -2$ or $1 < x < 2$ or $x > 4$.

As an alternative method you could have multiplied throughout by $(1-x)^2(x-4)^2$ (which is fine as we know that this is positive) and then solved $(1-x)(x-4)^2 < x(1-x)^2(x-4)$ which is equivalent to $(x-1)(x-4)(x^2-4) > 0$. There are quite a few places you can make sign errors along the way, and I would still draw a graph to solve this last inequality!

Example 2: $|x-2|+2 \geqslant |2x+1|$

Again, start by drawing two graphs. When $x < 2$ then $|x-2|+2 = -(x-2)+2 = 4-x$ and when $x > 2$ we have $|x-2|+2 = (x-2)+2 = x$. Similarly for the other graph.



To find one point of intersection you need to solve $-(x-2)+2 = -(2x+1)$ (giving $x = -5$) and for the other you need to solve $-(x-2)+2 = 2x+1$ (giving $x = 1$). We then want to find where the red line is above (or on) the blue line so we have $-5 \leqslant x \leqslant 1$.