STEP Support Programme

STEP 3 Polar Coordinates and other Coordinate Geometry: Topic Notes

Polar Coordinates

A point can be described in terms of *Polar Coordinates* (r, θ) where r is the distance of the point from the origin and θ is the angle the line connecting the point to the origin makes with the positive x-axis (measured anticlockwise).

It is usually assumed that $r \geqslant 0$, and you are usually told the range of θ as is appropriate for the particular question.

If Point P has Cartesian coordinates (x, y) and polar coordinates (r, θ) then:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Note that we also have $x^2 + y^2 = r^2$ and $\tan\theta = \frac{y}{x}$.

The **area of a sector** bounded by a curve and the half-lines $\theta = \alpha$ and $\theta = \beta$ is given by:

$$A = \tfrac{1}{2}\int_{\alpha}^{\beta} r^2 \mathrm{d}\theta$$

Conic Sections

The circle, ellipse, parabola and hyperbola are all types of "Conic Section" (although it can be argued that the circle is a special ellipse). Detailed knowledge of these (apart from the circle) is not assumed in the 2019 specifications but some of the following notes may be useful when working through past papers.

Conic sections are the curves you get at the intersections of a plane with a "right double cone". There is a nice diagram showing the curves here.

A circle is obtained when you slice the cone at right angles to the line of symmetry, a parabola is obtained if you slice parallel to the edge of the cone and a hyperbola intersects both parts of the "double cone". A *rectangular* hyperbola is one where the *asymptotes* are perpendicular.

These curves can be described in terms of a *focus*, *directrix* and *eccentricity*. The curve is the locus of all points whose distance to the *focus* F is a constant multiple (the *eccentricity*, e) of the distance of the point from a fixed line, L (the *directrix*).

If $0 < e < 1$ the curve is an ellipse, if $e = 1$ the curve is a parabola and if $e > 1$ the curve is a hyperbola. The circle is the limiting case of an ellipse as $e \to 0$.

Standard Forms

Circle

- Cartesian Equation: $x^2 + y^2 = a^2$
- Parametric Equations: $x = a\cos\theta$, $y = a\sin\theta$
- Intersection with the axes: $(\pm a, 0)$, $(0, \pm a)$

Ellipse

- Cartesian Equation: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
- Parametric Equations: $x = a\cos\theta$, $y = b\sin\theta$
- Intersection with the axes: $(\pm a, 0)$, $(0, \pm b)$

Parabola

- Cartesian Equation: $y^2 = 4ax$ where $a > 0$
- Parametric Equations: $x = at^2$, $y = 2at$
- Intersection with the axes: $(0, 0)$

Hyperbola

- Cartesian Equation: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
- Parametric Equations: $x = a\sec\theta$, $y = b\tan\theta$ **or** $x = \pm a\cosh u$, $y = b\sinh u$
- Intersection with the axes: $(\pm a, 0)$
- Asymptotes: $y = \pm\frac{b}{a}x$

Rectangular Hyperbola

- Cartesian Equation: $xy = c^2$
- Parametric Equations: $x = ct$, $y = \frac{c}{t}$
- Intersection with the axes: NONE
- Asymptotes: x-axis and y-axis

Top Tips!

- As with so many areas of STEP, good clear diagrams can make your life much easier when working on Geometry. Don't be afraid to draw several diagrams for different parts of a question.
- Coordinate Geometry questions in STEP 3 will expect that you can manipulate equations of straight lines, circles and parabolas, as well as performing calculus techniques to find areas or gradients. This can mean that once you've managed to draw the right diagram and model the situation, the question becomes more about performing lots of algebra than considering the geometry. Be careful though - don't just blindly apply algebraic techniques and forget for example that lengths are positive!