

STEP Support Programme

STEP 3 Hyperbolic Functions Questions

1 2004 S3 Q1

Show that

$$\int_0^a \frac{\sinh x}{2 \cosh^2 x - 1} dx = \frac{1}{2\sqrt{2}} \ln \left(\frac{\sqrt{2} \cosh a - 1}{\sqrt{2} \cosh a + 1} \right) + \frac{1}{2\sqrt{2}} \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)$$

and find

$$\int_0^a \frac{\cosh x}{1 + 2 \sinh^2 x} dx.$$

Hence show that

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2 \sinh^2 x} dx = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right).$$

By substituting $u = e^x$ in this result, or otherwise, find

$$\int_1^\infty \frac{1}{1 + u^4} du.$$

You might like to look at [2010 STEP 3 Question 2](#), which is similar to this one.

2 2011 S3 Q6

The definite integrals T , U , V and X are defined by

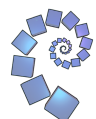
$$T = \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t} dt,$$

$$U = \int_{\ln 2}^{\ln 3} \frac{u}{2 \sinh u} du,$$

$$V = - \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln v}{1 - v^2} dv,$$

$$X = \int_{\frac{1}{2} \ln 2}^{\frac{1}{2} \ln 3} \ln(\coth x) dx.$$

Show, without evaluating any of them, that T , U , V and X are all equal.



3 2007 S3 Q5

Let $y = \ln(x^2 - 1)$, where $x > 1$, and let r and θ be functions of x determined by $r = \sqrt{x^2 - 1}$ and $\coth \theta = x$. Show that

$$\frac{dy}{dx} = \frac{2 \cosh \theta}{r} \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{2 \cosh 2\theta}{r^2},$$

and find an expression in terms of r and θ for $\frac{d^3y}{dx^3}$.

Find, with proof, a similar formula for $\frac{d^n y}{dx^n}$ in terms of r and θ .

4 2005 S3 Q6

In this question, you may use without proof the results

$$4 \cosh^3 y - 3 \cosh y = \cosh(3y) \quad \text{and} \quad \operatorname{arcosh} y = \ln(y + \sqrt{y^2 - 1}).$$

[**Note:** $\operatorname{arcosh} y$ is another notation for $\cosh^{-1} y$]

Show that the equation $x^3 - 3a^2x = 2a^3 \cosh T$ is satisfied by $2a \cosh\left(\frac{1}{3}T\right)$ and hence that, if $c^2 \geq b^3 > 0$, one of the roots of the equation $x^3 - 3bx = 2c$ is $u + \frac{b}{u}$, where $u = (c + \sqrt{c^2 - b^3})^{\frac{1}{3}}$.

Show that the other two roots of the equation $x^3 - 3bx = 2c$ are the roots of the quadratic equation

$$x^2 + \left(u + \frac{b}{u}\right)x + u^2 + \frac{b^2}{u^2} - b = 0,$$

and find these roots in terms of u , b and ω , where $\omega = \frac{1}{2}(-1 + i\sqrt{3})$.

Solve completely the equation $x^3 - 6x = 6$.

