STEP Support Programme

STEP 3 Hyperbolic Functions Questions

1 **2004 S3 Q1**

Show that

$$\int_0^a \frac{\sinh x}{2\cosh^2 x - 1}\,\mathrm{d}x = \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) + \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

and find

$$\int_0^a \frac{\cosh x}{1 + 2\sinh^2 x}\,\mathrm{d}x\,.$$

Hence show that

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2\sinh^2 x}\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\,.$$

By substituting $u = \mathrm{e}^x$ in this result, or otherwise, find

$$\int_1^\infty \frac{1}{1+u^4}\,\mathrm{d}u\,.$$

You might like to look at 2014 STEP 3 Question 2, which is similar to this one.

2 **2011 S3 Q6**

The definite integrals T, U, V and X are defined by

$$T = \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t}\,\mathrm{d}t\,, \qquad U = \int_{\ln 2}^{\ln 3} \frac{u}{2\sinh u}\,\mathrm{d}u\,,$$

$$V = -\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln v}{1 - v^2}\,\mathrm{d}v\,, \qquad X = \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln(\coth x)\,\mathrm{d}x\,.$$

Show, without evaluating any of them, that T, U, V and X are all equal.

3 **2007 S3 Q5**

Let $y = \ln(x^2 - 1)$, where $x > 1$, and let r and θ be functions of x determined by $r = \sqrt{x^2 - 1}$ and $\coth\theta = x$. Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cosh\theta}{r} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2\cosh 2\theta}{r^2},$$

and find an expression in terms of r and θ for $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$.

Find, with proof, a similar formula for $\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}$ in terms of r and θ.

4 **2005 S3 Q6**

In this question, you may use without proof the results

$$4\cosh^3 y - 3\cosh y = \cosh(3y) \quad \text{and} \quad \operatorname{arcosh} y = \ln(y + \sqrt{y^2 - 1}).$$

[**Note:** arcosh$y$ is another notation for $\cosh^{-1} y$]

Show that the equation $x^3 - 3a^2x = 2a^3\cosh T$ is satisfied by $2a\cosh\left(\frac{1}{3}T\right)$ and hence that, if $c^2 \geqslant b^3 > 0$, one of the roots of the equation $x^3 - 3bx = 2c$ is $u + \dfrac{b}{u}$, where $u = (c + \sqrt{c^2 - b^3})^{\frac{1}{3}}$.

Show that the other two roots of the equation $x^3 - 3bx = 2c$ are the roots of the quadratic equation

$$x^2 + \left(u + \frac{b}{u}\right)x + u^2 + \frac{b^2}{u^2} - b = 0,$$

and find these roots in terms of u, b and ω, where $\omega = \frac{1}{2}(-1 + \mathrm{i}\sqrt{3})$.

Solve completely the equation $x^3 - 6x = 6$.