

## STEP Support Programme

### STEP 3 Hyperbolic Functions: Solutions

1 Start by using the substitution  $t = \cosh x$ . This gives:

$$\begin{aligned}
 \int_0^a \frac{\sinh x}{2 \cosh^2 x - 1} dx &= \int_1^{\cosh a} \frac{\overline{\sinh x}}{2t^2 - 1} \times \frac{1}{\overline{\sinh x}} dt \\
 &= \frac{1}{2} \int_1^{\cosh a} \left( \frac{1}{\sqrt{2t-1}} - \frac{1}{\sqrt{2t+1}} \right) dt \\
 &= \frac{1}{2} \left[ \frac{1}{\sqrt{2}} \ln |\sqrt{2t-1}| - \frac{1}{\sqrt{2}} \ln |\sqrt{2t+1}| \right]_1^{\cosh a} \\
 &= \frac{1}{2\sqrt{2}} \left[ \ln (\sqrt{2} \cosh a - 1) - \ln (\sqrt{2} \cosh a + 1) \right] \\
 &\quad - \frac{1}{2\sqrt{2}} \left[ \ln (\sqrt{2} - 1) - \ln (\sqrt{2} + 1) \right] \\
 &= \frac{1}{2\sqrt{2}} \ln \left( \frac{\sqrt{2} \cosh a - 1}{\sqrt{2} \cosh a + 1} \right) + \frac{1}{2\sqrt{2}} \ln \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)
 \end{aligned}$$

Since the question said “show that” you should show how each stage is derived.

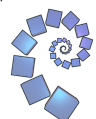
For the next integral use  $t = \sinh x$ . This gives:

$$\begin{aligned}
 \int_0^a \frac{\cosh x}{1 + 2 \sinh^2 x} dx &= \int_0^{\sinh a} \frac{\overline{\cosh x}}{1 + 2t^2} \times \frac{1}{\overline{\cosh x}} dt \\
 &= \frac{1}{2} \int_0^{\sinh a} \frac{1}{\frac{1}{2} + t^2} dt \\
 &= \frac{1}{2} \times \sqrt{2} \left[ \tan^{-1} (\sqrt{2}t) \right]_0^{\sinh a} \\
 &= \frac{1}{\sqrt{2}} \tan^{-1} (\sqrt{2} \sinh a)
 \end{aligned}$$

For the “Hence”, first note that  $1 + 2 \sinh^2 x = 1 + 2 (\cosh^2 x - 1) = 2 \cosh^2 x - 1$ . We then have:

$$\begin{aligned}
 \int_0^a \frac{\cosh x - \sinh x}{1 + 2 \sinh^2 x} dx &= \int_0^a \frac{\cosh x}{1 + 2 \sinh^2 x} dx - \int_0^a \frac{\sinh x}{1 + 2 \sinh^2 x} dx \\
 &= \int_0^a \frac{\cosh x}{1 + 2 \sinh^2 x} dx - \int_0^a \frac{\sinh x}{2 \cosh^2 x - 1} dx \\
 &= \frac{1}{\sqrt{2}} \tan^{-1} (\sqrt{2} \sinh a) - \frac{1}{2\sqrt{2}} \ln \left( \frac{\sqrt{2} \cosh a - 1}{\sqrt{2} \cosh a + 1} \right) - \frac{1}{2\sqrt{2}} \ln \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)
 \end{aligned}$$

Then as  $a \rightarrow \infty$  we have  $\cosh a \rightarrow \infty$  and  $\sinh a \rightarrow \infty$ . This means that  $\frac{\sqrt{2} \cosh a - 1}{\sqrt{2} \cosh a + 1} \rightarrow 1$



and  $\tan^{-1}(\sqrt{2} \sinh a) \rightarrow \frac{\pi}{2}$ . Hence we have:

$$\int_0^{\infty} \frac{\cosh x - \sinh x}{1 + 2 \sinh^2 x} dx = \frac{1}{\sqrt{2}} \times \frac{\pi}{2} - \frac{1}{2\sqrt{2}} \ln \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)$$

as required.

For the last part, start by noting that  $\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}(u + \frac{1}{u})$  and similarly  $\sinh x = \frac{1}{2}(u - \frac{1}{u})$ .

Using the substitution  $u = e^x$  gives:

$$\begin{aligned} \int_0^{\infty} \frac{\cosh x - \sinh x}{1 + 2 \sinh^2 x} dx &= \int_1^{\infty} \left[ \frac{1}{2} \left( u + \frac{1}{u} \right) - \frac{1}{2} \left( u - \frac{1}{u} \right) \right] \times \frac{1}{u + \frac{1}{2} \left( u^2 - 2 + \frac{1}{u^2} \right)} \times \frac{1}{u} du \\ &= \int_1^{\infty} \frac{2}{u} \times \frac{1}{u^2 + \frac{1}{u^2}} \times \frac{1}{u} du \\ &= 2 \int_1^{\infty} \frac{1}{u^4 + 1} du \end{aligned}$$

Hence from the previous result we have:

$$\int_1^{\infty} \frac{1}{u^4 + 1} du = \frac{\pi}{4\sqrt{2}} - \frac{1}{4\sqrt{2}} \ln \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right).$$



- 2 The hints document gives some useful formulae. There are lots of different approaches, this is just one possible method.

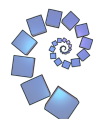
Using integration by parts on  $T$  gives:

$$\begin{aligned}
 \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t} dt &= \left[ \ln t \times \operatorname{artanh} t \right]_{\frac{1}{3}}^{\frac{1}{2}} - \int_{\frac{1}{3}}^{\frac{1}{2}} \left( \ln t \times \frac{1}{1-t^2} \right) dt && \text{using (6)} \\
 &= \left[ \ln t \times \frac{1}{2} \ln \left( \frac{1+t}{1-t} \right) \right]_{\frac{1}{3}}^{\frac{1}{2}} - \int_{\frac{1}{3}}^{\frac{1}{2}} \left( \ln t \times \frac{1}{1-t^2} \right) dt && \text{using (3)} \\
 &= \frac{1}{2} \left[ \ln \left( \frac{1}{2} \right) \times \ln \left( \frac{1+\frac{1}{2}}{1-\frac{1}{2}} \right) - \ln \left( \frac{1}{3} \right) \times \ln \left( \frac{1+\frac{1}{3}}{1-\frac{1}{3}} \right) \right] - \int_{\frac{1}{3}}^{\frac{1}{2}} \left( \ln t \times \frac{1}{1-t^2} \right) dt \\
 &= \frac{1}{2} \left[ \ln \left( \frac{1}{2} \right) \times \ln(3) - \ln \left( \frac{1}{3} \right) \times \ln(2) \right] - \int_{\frac{1}{3}}^{\frac{1}{2}} \left( \ln t \times \frac{1}{1-t^2} \right) dt \\
 &= \frac{1}{2} \left[ -\ln(2) \times \ln(3) + \ln(3) \times \ln(2) \right] - \int_{\frac{1}{3}}^{\frac{1}{2}} \left( \ln t \times \frac{1}{1-t^2} \right) dt \\
 &= 0 - \int_{\frac{1}{3}}^{\frac{1}{2}} \left( \ln t \times \frac{1}{1-t^2} \right) dt \\
 &= V
 \end{aligned}$$

Comparing  $U$  with  $T$  and  $V$ , it would be nice if I could convert a limit of  $\ln 2$  to one of  $\frac{1}{2}$ . If  $u = \ln 2$  implies  $t = \frac{1}{2}$  then it might be worth trying  $t = e^{-u}$ , which gives  $\frac{dt}{du} = -e^{-u} = -t$ . Using this substitution:

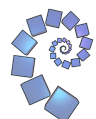
$$\begin{aligned}
 \int_{\ln 2}^{\ln 3} \frac{u}{2 \sinh u} du &= \int_{\ln 2}^{\ln 3} \frac{u}{e^u - e^{-u}} du \\
 &= \int_{\frac{1}{2}}^{\frac{1}{3}} \frac{-\ln t}{\frac{1}{t} - t} \times \frac{-1}{t} dt \\
 &= \int_{\frac{1}{2}}^{\frac{1}{3}} \frac{\ln t}{1-t^2} dt \\
 &= - \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln t}{1-t^2} dt \\
 &= V
 \end{aligned}$$

The final thing we need to do is show that  $X$  is equal to one of the other three. Looking at the limits  $t = \frac{1}{3}$  and  $x = \frac{1}{2} \ln 3$  suggests that we might want to use a substitution of  $x = \frac{1}{2} \ln \left( \frac{1}{t} \right) = -\frac{1}{2} \ln t$  or equivalently  $t = e^{-2x}$ .



Starting with  $T$  we have:

$$\begin{aligned}
 \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t} dt &= \int_{-\frac{1}{2} \ln(\frac{1}{3})}^{-\frac{1}{2} \ln(\frac{1}{2})} \frac{\operatorname{artanh}(e^{-2x})}{e^{-2x}} \times -2e^{-2x} dx \\
 &= -2 \int_{\frac{1}{2} \ln 3}^{\frac{1}{2} \ln 2} \operatorname{artanh}(e^{-2x}) dx \\
 &= \cancel{2} \int_{\frac{1}{2} \ln 2}^{\frac{1}{2} \ln 3} \cancel{\frac{1}{2}} \ln \left( \frac{1+e^{-2x}}{1-e^{-2x}} \right) dx \quad \text{using (3)} \\
 &= \int_{\frac{1}{2} \ln 2}^{\frac{1}{2} \ln 3} \ln \left( \frac{e^x + e^{-x}}{e^x - e^{-x}} \right) dx \\
 &\quad \text{multiplying top and bottom of fraction by } e^x \\
 &= \int_{\frac{1}{2} \ln 2}^{\frac{1}{2} \ln 3} \ln \left( \frac{\cosh x}{\sinh x} \right) dx \\
 &= \int_{\frac{1}{2} \ln 2}^{\frac{1}{2} \ln 3} \ln(\coth x) dx \\
 &= X
 \end{aligned}$$



**3** Differentiating we have  $\frac{dy}{dx} = \frac{1}{x^2 - 1} \times 2x$ , but this doesn't look immediately promising.

We also have:

$$y = \ln r^2 = 2 \ln r$$

$$\frac{dr}{dx} = x(x^2 - 1)^{\frac{1}{2}} = \frac{x}{\sqrt{x^2 - 1}}$$

Since we have  $\coth \theta = x$ ,  $x^2 - 1 = \coth^2 \theta - 1 = \operatorname{cosech}^2 \theta$ . Hence:

$$\begin{aligned} \frac{dr}{dx} &= \frac{x}{\sqrt{x^2 - 1}} \\ &= \frac{\coth \theta}{\operatorname{cosech} \theta} \\ &= \frac{\cosh \theta}{\sinh \theta} \times \sinh \theta \\ &= \cosh \theta \end{aligned}$$

We therefore have:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dr} \times \frac{dr}{dx} \\ &= \frac{2}{r} \times \cosh \theta \\ &= \frac{2 \cosh \theta}{r} \quad \text{as required} \end{aligned}$$

Now we differentiate again:

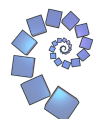
$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \left( \frac{2 \cosh \theta}{r} \right) \\ &= \frac{r \times \frac{d}{dx} (2 \cosh \theta) - 2 \cosh \theta \frac{dr}{dx}}{r^2} \\ &= \frac{r \times 2 \sinh \theta \frac{d\theta}{dx} - 2 \cosh \theta \frac{dr}{dx}}{r^2} \end{aligned}$$

Since  $x = \coth \theta$ , we have

$$\begin{aligned} \frac{dx}{d\theta} &= \frac{d}{d\theta} \left( \frac{\cosh \theta}{\sinh \theta} \right) \\ &= \frac{\sinh^2 \theta - \cosh^2 \theta}{\sinh^2 \theta} \\ &= \frac{-1}{\sinh^2 \theta} \\ &= -\operatorname{cosech}^2 \theta \end{aligned}$$

and so  $\frac{d\theta}{dx} = -\sinh^2 \theta$ . We also have  $r = \sqrt{\coth^2 \theta - 1} = \operatorname{cosech} \theta$  and  $\frac{dr}{dx} = \cosh \theta$ .

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{r \times 2 \sinh \theta \frac{d\theta}{dx} - 2 \cosh \theta \frac{dr}{dx}}{r^2} \\ &= \frac{-2 \sinh^2 \theta - 2 \cosh^2 \theta}{r^2} \\ &= -\frac{2 \cosh 2\theta}{r^2} \end{aligned}$$



Differentiating again gives:

$$\begin{aligned}
 \frac{d^3y}{dx^3} &= \frac{d}{dx} \left( -\frac{2 \cosh 2\theta}{r^2} \right) \\
 &= - \left( \frac{r^2 \times \frac{d}{dx} (2 \cosh 2\theta) - 2 \cosh 2\theta \times \frac{d}{dx} (r^2)}{r^4} \right) \\
 &= - \left( \frac{r^2 \times 4 \sinh 2\theta \frac{d\theta}{dx} - 2 \cosh 2\theta \times 2r \frac{dr}{dx}}{r^4} \right) \\
 &= - \left( \frac{r \times 4 \sinh 2\theta \frac{d\theta}{dx} - 2 \cosh 2\theta \times 2 \frac{dr}{dx}}{r^3} \right) \quad \text{cancelling } r \\
 &= - \left( \frac{\cancel{\cosh \theta} \times 4 \sinh 2\theta \times (-\sinh^2 \theta) - 2 \cosh 2\theta \times 2 \cosh \theta}{r^3} \right) \\
 &= \frac{4}{r^3} (\sinh 2\theta \sinh \theta + \cosh 2\theta \cosh \theta) \\
 &= \frac{4 \cosh 3\theta}{r^3}
 \end{aligned}$$

Looking at these results, a reasonable conjecture would be  $\frac{d^n y}{dx^n} = (-1)^{n-1} \frac{\text{something} \times \cosh n\theta}{r^n}$ .

To find a suitable expression for “something”, look back to see how these constants were formed previously. It might be helpful to look at what  $\frac{d^4 y}{dx^4}$  might be. If you differentiated again, the 4 would be multiplied by 3 (from both the power of  $r$  and the multiple of  $\theta$ ).

Hence we seem to have:

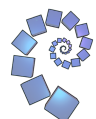
$n = 1$	constant = 2
$n = 2$	constant = $2 \times 1 = 2$
$n = 3$	constant = $2 \times 1 \times 2 = 4$
$n = 4$	constant = $2 \times 1 \times 2 \times 3 = 12$

So the “something” might be  $2(n-1)!$ .

Now we need to carry out the proof by induction.

**Conjecture:**  $\frac{d^n y}{dx^n} = 2(n-1)!(-1)^{n-1} \frac{\cosh n\theta}{r^n}$

**Base case:** From the previous work, we can see that the conjecture is true for  $n = 1, 2, 3$ .



**Inductive step:**

Assume the conjecture is true when  $n = k$ , so we have  $\frac{d^k y}{dx^k} = 2(k-1)!(-1)^{k-1} \frac{\cosh k\theta}{r^k}$ .

Differentiating with respect to  $x$  gives:

$$\begin{aligned}
 \frac{d^{(k+1)} y}{dx^{(k+1)}} &= 2(k-1)!(-1)^{k-1} \left( \frac{r^k \times k \sinh k\theta \frac{d\theta}{dx} - \cosh k\theta \times k r^{k-1} \frac{dr}{dx}}{r^{2k}} \right) \\
 &= 2(k-1)!(-1)^{k-1} \left[ \frac{k \times r^{k-1}}{r^{2k}} \left( r \sinh k\theta \frac{d\theta}{dx} - \cosh k\theta \frac{dr}{dx} \right) \right] \\
 &= 2(k-1)!(-1)^{k-1} \left[ \frac{k}{r^{k+1}} (\operatorname{cosech} \theta \sinh k\theta \times (-\sinh^2 \theta) - \cosh k\theta \times \cosh \theta) \right] \\
 &= 2(k-1)!(-1)^{k-1} \left[ \frac{k}{r^{k+1}} (-1) (\sinh k\theta \times \sinh \theta + \cosh k\theta \times \cosh \theta) \right] \\
 &= 2k \times (k-1)!(-1) \times (-1)^{k-1} \frac{\cosh(k+1)\theta}{r^{k+1}} \\
 &= 2(k)!(-1)^k \frac{\cosh(k+1)\theta}{r^{k+1}}
 \end{aligned}$$

Which is the same expression as the conjecture with  $n = k + 1$ .

Hence the conjecture is true for  $n = k$  then it is true for  $n = k + 1$ , and since it is true for  $n = 1$  it is true for all integers  $n \geq 1$ .



**4** This is quite a long question!

Substituting  $x = 2a \cosh\left(\frac{1}{3}T\right)$  into the left hand side of the equation gives:

$$\begin{aligned} x^3 - 3a^2x &= 8a^3 \cosh^3\left(\frac{1}{3}T\right) - 6a^3 \cosh\left(\frac{1}{3}T\right) \\ &= 2a^3 \left(4 \cosh^2\left(\frac{1}{3}T\right) - 3 \cosh\left(\frac{1}{3}T\right)\right) \\ &= 2a^3 \cosh T \quad \text{using the first given result} \end{aligned}$$

Hence  $x = 2a \cosh\left(\frac{1}{3}T\right)$  is a solution to the equation.

Comparing  $x^3 - 3bx = 2c$  and  $x^3 - 3a^2x = 2a^3 \cosh T$  it appears that we want to take  $b = a^2$  (which as  $b^3 > 0 \implies b > 0$  is an ok thing to do).

Further we want  $c = a^3 \cosh T$  i.e.  $\cosh T = \frac{c}{a^3}$ . For this to be ok we need  $\frac{c}{a^3} \geq 1$ . We are told that  $c^2 \geq b^3$ , and as we are taking  $b = a^2$  this means  $c^2 \geq a^6$ . As long as  $c$  and  $a$  have the same sign, this means that  $c \geq a^3$  and  $\frac{c}{a^3} \geq 1$ .

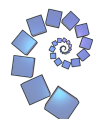
We therefore know that one solution is  $x = 2a \cosh\left(\frac{T}{3}\right)$ .

Using the second result given at the start of the question we have:

$$\begin{aligned} T &= \operatorname{arcosh}\left(\frac{c}{a^3}\right) \\ &= \ln\left(\frac{c}{a^3} + \sqrt{\frac{c^2}{a^6} - 1}\right) \\ &= \ln\left(\frac{c + \sqrt{c^2 - a^6}}{a^3}\right) \\ &= \ln\left(\frac{c + \sqrt{c^2 - b^3}}{a^3}\right) \\ &= \ln\left(\frac{u^3}{a^3}\right) \\ &= 3 \ln\left(\frac{u}{a}\right) \end{aligned}$$

Therefore the root becomes:

$$\begin{aligned} x &= 2a \cosh\left(\frac{T}{3}\right) \\ &= 2a \times \cosh\left(\ln\left(\frac{u}{a}\right)\right) \\ &= 2a \times \frac{1}{2} \left(e^{\ln\left(\frac{u}{a}\right)} + e^{-\ln\left(\frac{u}{a}\right)}\right) \\ &= a \times \left(\frac{u}{a} + \frac{a}{u}\right) \\ &= u + \frac{a^2}{u} \\ &= u + \frac{b}{u} \end{aligned}$$





We now have a root  $x = u + \frac{b}{u}$ , which means that  $\left(x - u - \frac{b}{u}\right)$  is a factor of  $x^3 - 3bx - 2c$ .

There are various ways to proceed, including long division or by using:

$$x^3 - 3bx - 2c \equiv \left(x - u - \frac{b}{u}\right) (x^2 + Ax + B)$$

Equating coefficients for this last one gives us:

$$B = 2c \div \left(u + \frac{b}{u}\right)$$

$$A = u + \frac{b}{u}$$

and so the other roots are the roots of the equation:

$$x^2 + \left(u + \frac{b}{u}\right)x + \left[2c \div \left(u + \frac{b}{u}\right)\right] = 0.$$

This is not in the required form yet, but since  $x = u + \frac{b}{u}$  is a solution to  $x^3 - 3bx = 2c$  we have:

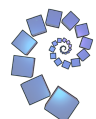
$$\begin{aligned} 2c &= \left(u + \frac{b}{u}\right)^3 - 3b\left(u + \frac{b}{u}\right) \\ &= \left(u + \frac{b}{u}\right) \left[\left(u + \frac{b}{u}\right)^2 - 3b\right] \\ &= \left(u + \frac{b}{u}\right) \left[u^2 + 2b + \frac{b^2}{u^2} - 3b\right] \\ &= \left(u + \frac{b}{u}\right) \left[u^2 + \frac{b^2}{u^2} - b\right] \end{aligned}$$

So now we know that the other roots are the roots of the equation:

$$x^2 + \left(u + \frac{b}{u}\right)x + \left[u^2 + \frac{b^2}{u^2} - b\right] = 0.$$

Using the quadratic formula we have:

$$\begin{aligned} &\frac{1}{2} \left[ -\left(u + \frac{b}{u}\right) \pm \sqrt{\left(u + \frac{b}{u}\right)^2 - 4\left(u^2 + \frac{b^2}{u^2} - b\right)} \right] \\ &= \frac{1}{2} \left[ -\left(u + \frac{b}{u}\right) \pm \sqrt{u^2 + 2b + \frac{b^2}{u^2} - 4u^2 - 4\frac{b^2}{u^2} + 4b} \right] \\ &= \frac{1}{2} \left[ -\left(u + \frac{b}{u}\right) \pm \sqrt{-3\left(u^2 + \frac{b^2}{u^2} - 2b\right)} \right] \\ &= \frac{1}{2} \left[ -\left(u + \frac{b}{u}\right) \pm \sqrt{-3} \times \left(u - \frac{b}{u}\right) \right] \\ &= \frac{1}{2} \left[ -\left(u + \frac{b}{u}\right) \pm i\sqrt{3} \times \left(u - \frac{b}{u}\right) \right] \end{aligned}$$



We now want this in terms of  $\omega = \frac{1}{2}(-1 + i\sqrt{3})$ . We have:

$$\begin{aligned} \frac{1}{2} \left[ - \left( u + \frac{b}{u} \right) + i\sqrt{3} \times \left( u - \frac{b}{u} \right) \right] &= u \times \frac{1}{2}(-1 + i\sqrt{3}) + \frac{b}{u} \times \frac{1}{2}(-1 - i\sqrt{3}) \\ &= u\omega + \frac{b}{u}\omega^2 \end{aligned}$$

Noting that  $\omega^2 = \frac{1}{4}(1 - 3 - 2i\sqrt{3}) = \frac{1}{2}(-1 - i\sqrt{3})$ .

The other root is:

$$\begin{aligned} \frac{1}{2} \left[ - \left( u + \frac{b}{u} \right) - i\sqrt{3} \times \left( u - \frac{b}{u} \right) \right] &= u \times \frac{1}{2}(-1 - i\sqrt{3}) + \frac{b}{u} \times \frac{1}{2}(-1 + i\sqrt{3}) \\ &= u\omega^2 + \frac{b}{u}\omega \end{aligned}$$

For the final part, we have  $x^3 - 6x = 6$  which means  $b = 2$  and  $c = 3$ . This gives:

$$\begin{aligned} a &= \sqrt{b} = \sqrt{2} \\ u &= \left( c + \sqrt{c^2 - b^3} \right)^{\frac{1}{3}} = (3 + 1)^{\frac{1}{3}} = 2^{\frac{2}{3}} \\ \frac{b}{u} &= \frac{2}{2^{\frac{2}{3}}} = 2^{\frac{1}{3}} \end{aligned}$$

The solutions are therefore:

$$\begin{aligned} u + \frac{b}{u} &= 2^{\frac{2}{3}} + 2^{\frac{1}{3}} \\ u\omega + \frac{b}{u}\omega^2 &= 2^{\frac{2}{3}}\omega + 2^{\frac{1}{3}}\omega^2 \\ u\omega^2 + \frac{b}{u}\omega &= 2^{\frac{2}{3}}\omega^2 + 2^{\frac{1}{3}}\omega \end{aligned}$$

