STEP Support Programme

STEP 3 Hyperbolic Functions: Solutions

1 Start by using the substitution $t = \cosh x$. This gives:

$$\begin{aligned}
\int_0^a \frac{\sinh x}{2\cosh^2 x - 1}\,dx &= \int_1^{\cosh a} \frac{\cancel{\sinh x}}{2t^2-1} \times \frac{1}{\cancel{\sinh x}}\,dt \\
&= \tfrac{1}{2}\int_1^{\cosh a} \left(\frac{1}{\sqrt{2}t-1} - \frac{1}{\sqrt{2}t+1}\right) dt \\
&= \tfrac{1}{2}\left[\frac{1}{\sqrt{2}}\ln|\sqrt{2}t-1| - \frac{1}{\sqrt{2}}\ln|\sqrt{2}t+1|\right]_1^{\cosh a} \\
&= \frac{1}{2\sqrt{2}}\left[\ln\left(\sqrt{2}\cosh a - 1\right) - \ln\left(\sqrt{2}\cosh a + 1\right)\right] \\
&\qquad - \frac{1}{2\sqrt{2}}\left[\ln\left(\sqrt{2}-1\right) - \ln\left(\sqrt{2}+1\right)\right] \\
&= \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) + \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)
\end{aligned}$$

Since the question said "show that" you should show how each stage is derived.

For the next integral use $t = \sinh x$. This gives:

$$\begin{aligned}
\int_0^a \frac{\cosh x}{1 + 2\sinh^2 x}\,dx &= \int_0^{\sinh a} \frac{\cancel{\cosh x}}{1+2t^2} \times \frac{1}{\cancel{\cosh x}}\,dt \\
&= \tfrac{1}{2}\int_0^{\sinh a} \frac{1}{\frac{1}{2}+t^2}\,dt \\
&= \tfrac{1}{2} \times \sqrt{2}\left[\tan^{-1}\left(\sqrt{2}t\right)\right]_0^{\sinh a} \\
&= \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\sinh a\right)
\end{aligned}$$

For the "Hence", first note that $1 + 2\sinh^2 x = 1 + 2\left(\cosh^2 x - 1\right) = 2\cosh^2 x - 1$. We then have:

$$\begin{aligned}
\int_0^a \frac{\cosh x - \sinh x}{1 + 2\sinh^2 x}\,dx &= \int_0^a \frac{\cosh x}{1+2\sinh^2 x}\,dx - \int_0^a \frac{\sinh x}{1 + 2\sinh^2 x}\,dx \\
&= \int_0^a \frac{\cosh x}{1+2\sinh^2 x}\,dx - \int_0^a \frac{\sinh x}{2\cosh^2 x - 1}\,dx \\
&= \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\sinh a\right) - \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) - \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)
\end{aligned}$$

Then as $a \to \infty$ we have $\cosh a \to \infty$ and $\sinh a \to \infty$. This means that $\dfrac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1} \to 1$

and $\tan^{-1}\left(\sqrt{2}\sinh a\right) \to \dfrac{\pi}{2}$. Hence we have:

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2\sinh^2 x}\,\mathrm{d}x = \frac{1}{\sqrt{2}} \times \frac{\pi}{2} - \frac{1}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

as required.

For the last part, start by noting that $\cosh x = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right) = \frac{1}{2}\left(u + \frac{1}{u}\right)$ and similarly $\sinh x = \frac{1}{2}\left(u - \frac{1}{u}\right)$.
Using the substitution $u = \mathrm{e}^x$ gives:

$$\begin{aligned}\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2\sinh^2 x}\,\mathrm{d}x &= \int_1^\infty \left[\tfrac{1}{2}\left(\cancel{u} + \frac{1}{u}\right) - \tfrac{1}{2}\left(\cancel{u} - \frac{1}{u}\right)\right] \times \frac{1}{\cancel{1} + \frac{1}{2}\left(u^2 - \cancel{2} + \frac{1}{u^2}\right)} \times \frac{1}{u}\,\mathrm{d}u \\ &= \int_1^\infty \frac{2}{u} \times \frac{1}{u^2 + \frac{1}{u^2}} \times \frac{1}{u}\,\mathrm{d}u \\ &= 2\int_1^\infty \frac{1}{u^4+1}\,\mathrm{d}u\end{aligned}$$

Hence from the previous result we have:

$$\int_1^\infty \frac{1}{u^4+1}\,\mathrm{d}u = \frac{\pi}{4\sqrt{2}} - \frac{1}{4\sqrt{2}} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

2 The hints document gives some useful formulae. There are lots of different approaches, this is just one possible method.

Using integration by parts on T gives:

$$\begin{aligned}
\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t}\,\mathrm{d}t &= \Big[\ln t \times \operatorname{artanh} t\Big]_{\frac{1}{3}}^{\frac{1}{2}} - \int_{\frac{1}{3}}^{\frac{1}{2}} \left(\ln t \times \frac{1}{1-t^2}\right)\mathrm{d}t && \text{using (6)}\\
&= \left[\ln t \times \tfrac{1}{2}\ln\left(\frac{1+t}{1-t}\right)\right]_{\frac{1}{3}}^{\frac{1}{2}} - \int_{\frac{1}{3}}^{\frac{1}{2}} \left(\ln t \times \frac{1}{1-t^2}\right)\mathrm{d}t && \text{using (3)}\\
&= \frac{1}{2}\left[\ln\left(\tfrac{1}{2}\right)\times\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) - \ln\left(\tfrac{1}{3}\right)\times\ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)\right] - \int_{\frac{1}{3}}^{\frac{1}{2}} \left(\ln t \times \frac{1}{1-t^2}\right)\mathrm{d}t\\
&= \frac{1}{2}\left[\ln\left(\tfrac{1}{2}\right)\times\ln(3) - \ln\left(\tfrac{1}{3}\right)\times\ln(2)\right] - \int_{\frac{1}{3}}^{\frac{1}{2}} \left(\ln t \times \frac{1}{1-t^2}\right)\mathrm{d}t\\
&= \frac{1}{2}\left[-\ln(2)\times\ln(3) + \ln(3)\times\ln(2)\right] - \int_{\frac{1}{3}}^{\frac{1}{2}} \left(\ln t \times \frac{1}{1-t^2}\right)\mathrm{d}t\\
&= 0 - \int_{\frac{1}{3}}^{\frac{1}{2}} \left(\ln t \times \frac{1}{1-t^2}\right)\mathrm{d}t\\
&= V
\end{aligned}$$

Comparing U with T and V, it would be nice if I could convert a limit of $\ln 2$ to one of $\frac{1}{2}$. If $u = \ln 2$ implies $t = \frac{1}{2}$ then it might be worth trying $t = \mathrm{e}^{-u}$, which gives $\dfrac{\mathrm{d}t}{\mathrm{d}u} = -\mathrm{e}^{-u} = -t$. Using this substitution:

$$\begin{aligned}
\int_{\ln 2}^{\ln 3} \frac{u}{2\sinh u}\,\mathrm{d}u &= \int_{\ln 2}^{\ln 3} \frac{u}{\mathrm{e}^u - \mathrm{e}^{-u}}\,\mathrm{d}u\\
&= \int_{\frac{1}{2}}^{\frac{1}{3}} \frac{-\ln t}{\frac{1}{t}-t} \times \frac{-1}{t}\,\mathrm{d}t\\
&= \int_{\frac{1}{2}}^{\frac{1}{3}} \frac{\ln t}{1-t^2}\,\mathrm{d}t\\
&= -\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln t}{1-t^2}\,\mathrm{d}t\\
&= V
\end{aligned}$$

The final thing we need to do is show that X is equal to one of the other three. Looking at the limits $t = \frac{1}{3}$ and $x = \frac{1}{2}\ln 3$ suggests that we might want to use a substitution of $x = \frac{1}{2}\ln\left(\frac{1}{t}\right) = -\frac{1}{2}\ln t$ or equivalently $t = \mathrm{e}^{-2x}$.

Starting with T we have:

$$\begin{aligned}
\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t}\,\mathrm{d}t &= \int_{-\frac{1}{2}\ln\left(\frac{1}{3}\right)}^{-\frac{1}{2}\ln\left(\frac{1}{2}\right)} \frac{\operatorname{artanh}\left(\mathrm{e}^{-2x}\right)}{\cancel{\mathrm{e}^{-2x}}} \times -2\cancel{\mathrm{e}^{-2x}}\,\mathrm{d}x \\
&= -2\int_{\frac{1}{2}\ln 3}^{\frac{1}{2}\ln 2} \operatorname{artanh}\left(\mathrm{e}^{-2x}\right)\,\mathrm{d}x \\
&= \cancel{2}\int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \cancel{\tfrac{1}{2}}\ln\left(\frac{1+\mathrm{e}^{-2x}}{1-\mathrm{e}^{-2x}}\right)\,\mathrm{d}x \qquad \textit{using } (3) \\
&= \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln\left(\frac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{\mathrm{e}^{x}-\mathrm{e}^{-x}}\right)\,\mathrm{d}x \\
&\qquad\qquad \textit{multiplying top and bottom of fraction by } \mathrm{e}^{x} \\
&= \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln\left(\frac{\cosh x}{\sinh x}\right)\,\mathrm{d}x \\
&= \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln\left(\coth x\right)\,\mathrm{d}x \\
&= X
\end{aligned}$$

3 Differentiating we have $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2-1} \times 2x$, but this doesn't look immediately promising.

We also have:

$$\begin{aligned} y &= \ln r^2 = 2\ln r \\ \frac{\mathrm{d}r}{\mathrm{d}x} &= x\left(x^2-1\right)^{\frac{1}{2}} = \frac{x}{\sqrt{x^2-1}} \end{aligned}$$

Since we have $\coth\theta = x$, $x^2 - 1 = \coth^2\theta - 1 = \operatorname{cosech}^2\theta$. Hence:

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}x} &= \frac{x}{\sqrt{x^2-1}} \\ &= \frac{\coth\theta}{\operatorname{cosech}\theta} \\ &= \frac{\cosh\theta}{\sinh\theta} \times \sinh\theta \\ &= \cosh\theta \end{aligned}$$

We therefore have:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}x} \\ &= \frac{2}{r} \times \cosh\theta \\ &= \frac{2\cosh\theta}{r} \qquad \textit{as required} \end{aligned}$$

Now we differentiate again:

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2\cosh\theta}{r}\right) \\ &= \frac{r \times \frac{\mathrm{d}}{\mathrm{d}x}(2\cosh\theta) - 2\cosh\theta\,\frac{\mathrm{d}r}{\mathrm{d}x}}{r^2} \\ &= \frac{r \times 2\sinh\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}x} - 2\cosh\theta\,\frac{\mathrm{d}r}{\mathrm{d}x}}{r^2} \end{aligned}$$

Since $x = \coth\theta$, we have

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\theta} &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\cosh\theta}{\sinh\theta}\right) \\ &= \frac{\sinh^2\theta - \cosh^2\theta}{\sinh^2\theta} \\ &= \frac{-1}{\sinh^2\theta} \\ &= -\operatorname{cosech}^2\theta \end{aligned}$$

and so $\frac{\mathrm{d}\theta}{\mathrm{d}x} = -\sinh^2\theta$. We also have $r = \sqrt{\coth^2\theta - 1} = \operatorname{cosech}\theta$ and $\frac{\mathrm{d}r}{\mathrm{d}x} = \cosh\theta$.

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\cancel{r} \times 2\cancel{\sinh\theta}\,\frac{\mathrm{d}\theta}{\mathrm{d}x} - 2\cosh\theta\,\frac{\mathrm{d}r}{\mathrm{d}x}}{r^2} \\ &= \frac{-2\sinh^2\theta - 2\cosh^2\theta}{r^2} \\ &= -\frac{2\cosh 2\theta}{r^2} \end{aligned}$$

Differentiating again gives:

$$\begin{aligned}
\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{2\cosh 2\theta}{r^2}\right) \\
&= -\left(\frac{r^2 \times \frac{\mathrm{d}}{\mathrm{d}x}(2\cosh 2\theta) - 2\cosh 2\theta \times \frac{\mathrm{d}}{\mathrm{d}x}\left(r^2\right)}{r^4}\right) \\
&= -\left(\frac{r^2 \times 4\sinh 2\theta \frac{\mathrm{d}\theta}{\mathrm{d}x} - 2\cosh 2\theta \times 2r\frac{\mathrm{d}r}{\mathrm{d}x}}{r^4}\right) \\
&= -\left(\frac{r \times 4\sinh 2\theta \frac{\mathrm{d}\theta}{\mathrm{d}x} - 2\cosh 2\theta \times 2\frac{\mathrm{d}r}{\mathrm{d}x}}{r^3}\right) \qquad \textit{cancelling } r \\
&= -\left(\frac{\cancel{\operatorname{cosech}\theta} \times 4\sinh 2\theta \times \left(-\sinh^{\cancel{2}}\theta\right) - 2\cosh 2\theta \times 2\cosh\theta}{r^3}\right) \\
&= \frac{4}{r^3}\left(\sinh 2\theta \sinh\theta + \cosh 2\theta \cosh\theta\right) \\
&= \frac{4\cosh 3\theta}{r^3}
\end{aligned}$$

Looking at these results, a reasonable conjecture would be $\frac{\mathrm{d}^ny}{\mathrm{d}x^2} = (-1)^{n-1}\frac{\text{something} \times \cosh n\theta}{r^n}$. To find a suitable expression for "something", look back to see how these constants were formed previously. It might be helpful to look at what $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}$ might be. If you differentiated again, the 4 would be multiplied by 3 (from both the power of r and the multiple of θ).

Hence we seem to have:

$$\begin{aligned}
n &= 1 \qquad \text{constant } &&= 2 \\
n &= 2 \qquad \text{constant } &&= 2 \times 1 = 2 \\
n &= 3 \qquad \text{constant } &&= 2 \times 1 \times 2 = 4 \\
n &= 4 \qquad \text{constant } &&= 2 \times 1 \times 2 \times 3 = 12
\end{aligned}$$

So the "something" might be $2(n-1)!$.

Now we need to carry out the proof by induction.

Conjecture: $\frac{\mathrm{d}^ny}{\mathrm{d}x^n} = 2(n-1)!(-1)^{n-1}\frac{\cosh n\theta}{r^n}$

Base case: From the previous work, we can see that the conjecture is true for $n = 1, 2, 3$.

Inductive step:

Assume the conjecture is true when $n = k$, so we have $\dfrac{\mathrm{d}^k y}{\mathrm{d}x^k} = 2(k-1)!(-1)^{k-1}\,\dfrac{\cosh k\theta}{r^k}$.

Differentiating with respect to x gives:

$$\begin{aligned}
\frac{\mathrm{d}^{(k+1)}\, y}{\mathrm{d}x^{(k+1)}} &= 2(k-1)!(-1)^{k-1}\left(\frac{r^k \times k\sinh k\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}x} - \cosh k\theta \times k r^{k-1}\,\frac{\mathrm{d}r}{\mathrm{d}x}}{r^{2k}}\right)\\
&= 2(k-1)!(-1)^{k-1}\left[\frac{k \times r^{k-1}}{r^{2k}}\left(r\sinh k\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}x} - \cosh k\theta\,\frac{\mathrm{d}r}{\mathrm{d}x}\right)\right]\\
&= 2(k-1)!(-1)^{k-1}\left[\frac{k}{r^{k+1}}\left(\operatorname{cosech}\theta\,\sinh k\theta \times \left(-\sinh^2\theta\right) - \cosh k\theta \times \cosh\theta\right)\right]\\
&= 2(k-1)!(-1)^{k-1}\left[\frac{k}{r^{k+1}}(-1)\left(\sinh k\theta \times \sinh\theta + \cosh k\theta \times \cosh\theta\right)\right]\\
&= 2\,k \times (k-1)!\,(-1) \times (-1)^{k-1}\,\frac{\cosh(k+1)\theta}{r^{k+1}}\\
&= 2(k)!(-1)^k\,\frac{\cosh(k+1)\theta}{r^{k+1}}
\end{aligned}$$

Which is the same expression as the conjecture with $n = k + 1$.

Hence the conjecture is true for $n = k$ then it is true for $n = k + 1$, and since it is true for $n = 1$ it is true for all integers $n \geqslant 1$.

4 This is quite a long question!

Substituting $x = 2a\cosh\left(\frac{1}{3}T\right)$ into the left hand side of the equation gives:

$$\begin{aligned} x^3 - 3a^2x &= 8a^3\cosh^3\left(\tfrac{1}{3}T\right) - 6a^3\cosh\left(\tfrac{1}{3}T\right) \\ &= 2a^3\left(4\cosh^2\left(\tfrac{1}{3}T\right) - 3\cosh\left(\tfrac{1}{3}T\right)\right) \\ &= 2a^3\cosh T \qquad \textit{using the first given result} \end{aligned}$$

Hence $x = 2a\cosh\left(\frac{1}{3}T\right)$ is a solution to the equation.

Comparing $x^3 - 3bx = 2c$ and $x^3 - 3a^2x = 2a^3\cosh T$ is appears that we want to take $b = a^2$ (which as $b^3 > 0 \implies b > 0$ is an ok thing to do).

Further we want $c = a^3\cosh T$ i.e. $\cosh T = \dfrac{c}{a^3}$. For this to be ok we need $\dfrac{c}{a^3} \geqslant 1$. We are told that $c^2 \geqslant b^3$, and as we are taking $b = a^2$ this means $c^2 \geqslant a^6$. As long as c and a have the same sign, this means that $c \geqslant a^3$ and $\dfrac{c}{a^3} \geqslant 1$.

We therefore know that one solution is $x = 2a\cosh\left(\dfrac{T}{3}\right)$.

Using the second result given at the start of the question we have:

$$\begin{aligned} T &= \operatorname{arcosh}\left(\frac{c}{a^3}\right) \\ &= \ln\left(\frac{c}{a^3} + \sqrt{\frac{c^2}{a^6} - 1}\right) \\ &= \ln\left(\frac{c + \sqrt{c^2 - a^6}}{a^3}\right) \\ &= \ln\left(\frac{c + \sqrt{c^2 - b^3}}{a^3}\right) \\ &= \ln\left(\frac{u^3}{a^3}\right) \\ &= 3\ln\left(\frac{u}{a}\right) \end{aligned}$$

Therefore the root becomes:

$$\begin{aligned} x &= 2a\cosh\left(\frac{T}{3}\right) \\ &= 2a \times \cosh\left(\ln\left(\frac{u}{a}\right)\right) \\ &= \cancel{2}a \times \frac{1}{\cancel{2}}\left(e^{\ln\left(\frac{u}{a}\right)} + e^{-\ln\left(\frac{u}{a}\right)}\right) \\ &= a \times \left(\frac{u}{a} + \frac{a}{u}\right) \\ &= u + \frac{a^2}{u} \\ &= u + \frac{b}{u} \end{aligned}$$

We now have a root $x = u + \dfrac{b}{u}$, which means that $\left(x - u - \dfrac{b}{u}\right)$ is a factor of $x^3 - 3bx - 2c$. There are various ways to proceed, including long division or by using:

$$x^3 - 3bx - 2c \equiv \left(x - u - \frac{b}{u}\right)\left(x^2 + Ax + B\right)$$

Equating coefficients for this last one gives us:

$$B = 2c \div \left(u + \frac{b}{u}\right)$$
$$A = u + \frac{b}{u}$$

and so the other roots are the roots of the equation:

$$x^2 + \left(u + \frac{b}{u}\right)x + \left[2c \div \left(u + \frac{b}{u}\right)\right] = 0\,.$$

This is not in the required form yet, but since $x = u + \dfrac{b}{u}$ is a solution to $x^3 - 3bx = 2c$ we have:

$$\begin{aligned}
2c &= \left(u + \frac{b}{u}\right)^3 - 3b\left(u + \frac{b}{u}\right) \\
&= \left(u + \frac{b}{u}\right)\left[\left(u + \frac{b}{u}\right)^2 - 3b\right] \\
&= \left(u + \frac{b}{u}\right)\left[u^2 + 2b + \frac{b^2}{u^2} - 3b\right] \\
&= \left(u + \frac{b}{u}\right)\left[u^2 + \frac{b^2}{u^2} - b\right]
\end{aligned}$$

So now we know that the other roots are the roots of the equation:

$$x^2 + \left(u + \frac{b}{u}\right)x + \left[u^2 + \frac{b^2}{u^2} - b\right] = 0\,.$$

Using the quadratic formula we have:

$$\begin{aligned}
&\tfrac{1}{2}\left[-\left(u + \frac{b}{u}\right) \pm \sqrt{\left(u + \frac{b}{u}\right)^2 - 4\left(u^2 + \frac{b^2}{u^2} - b\right)}\right] \\
=&\tfrac{1}{2}\left[-\left(u + \frac{b}{u}\right) \pm \sqrt{u^2 + 2b + \frac{b^2}{u^2} - 4u^2 - 4\frac{b^2}{u^2} + 4b}\right] \\
=&\tfrac{1}{2}\left[-\left(u + \frac{b}{u}\right) \pm \sqrt{-3\left(u^2 + \frac{b^2}{u^2} - 2b\right)}\right] \\
=&\tfrac{1}{2}\left[-\left(u + \frac{b}{u}\right) \pm \sqrt{-3} \times \left(u - \frac{b}{u}\right)\right] \\
=&\tfrac{1}{2}\left[-\left(u + \frac{b}{u}\right) \pm \mathrm{i}\sqrt{3} \times \left(u - \frac{b}{u}\right)\right]
\end{aligned}$$

We now want this in terms of $\omega = \frac{1}{2}\left(-1+\mathrm{i}\sqrt{3}\right)$. We have:

$$\begin{aligned}\tfrac{1}{2}\left[-\left(u+\frac{b}{u}\right)+\mathrm{i}\sqrt{3}\times\left(u-\frac{b}{u}\right)\right] &= u\times\tfrac{1}{2}\left(-1+\mathrm{i}\sqrt{3}\right)+\frac{b}{u}\times\tfrac{1}{2}\left(-1-\mathrm{i}\sqrt{3}\right)\\ &= u\omega+\frac{b}{u}\omega^2\end{aligned}$$

Noting that $\omega^2 = \frac{1}{4}(1-3-2\mathrm{i}\sqrt{3}) = \frac{1}{2}(-1-\mathrm{i}\sqrt{3})$.

The other root is:

$$\begin{aligned}\tfrac{1}{2}\left[-\left(u+\frac{b}{u}\right)-\mathrm{i}\sqrt{3}\times\left(u-\frac{b}{u}\right)\right] &= u\times\tfrac{1}{2}\left(-1-\mathrm{i}\sqrt{3}\right)+\frac{b}{u}\times\tfrac{1}{2}\left(-1+\mathrm{i}\sqrt{3}\right)\\ &= u\omega^2+\frac{b}{u}\omega\end{aligned}$$

For the final part, we have $x^3-6x=6$ which means $b=2$ and $c=3$. This gives:

$$\begin{aligned}a &= \sqrt{b}=\sqrt{2}\\ u &= \left(c+\sqrt{c^2-b^3}\right)^{\frac{1}{3}} = (3+1)^{\frac{1}{3}} = 2^{\frac{2}{3}}\\ \frac{b}{u} &= \frac{2}{2^{\frac{2}{3}}} = 2^{\frac{1}{3}}\end{aligned}$$

The solutions are therefore:

$$\begin{aligned}u+\frac{b}{u} &= 2^{\frac{2}{3}}+2^{\frac{1}{3}}\\ u\omega+\frac{b}{u}\omega^2 &= 2^{\frac{2}{3}}\omega+2^{\frac{1}{3}}\omega^2\\ u\omega^2+\frac{b}{u}\omega &= 2^{\frac{2}{3}}\omega^2+2^{\frac{1}{3}}\omega\end{aligned}$$