

STEP Support Programme

STEP 3 Hyperbolic Functions Topic Notes

Definitions

$$\begin{aligned}\cosh x &= \frac{1}{2}(e^x + e^{-x}) \\ \sinh x &= \frac{1}{2}(e^x - e^{-x}) \\ \tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ \operatorname{sech} x &= \frac{1}{\cosh x} \\ \operatorname{cosech} x &= \frac{1}{\sinh x} \\ \operatorname{coth} x &= \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}\end{aligned}$$

Graphs and ranges

The graphs can be sketched by using your understanding of e^x , odd/even functions etc. For example:

$\cosh x$ is an even function (as $\cosh(-x) = \cosh x$) and so it is symmetrical in the y -axis. It passes through $(0, 1)$ and as $x \rightarrow \pm\infty$, $y \rightarrow \infty$. You can also use [Desmos](#) to sketch these.

We have:

$$\begin{aligned}1 &\leq \cosh x < \infty \\ -\infty &< \sinh x < \infty \\ -1 &< \tanh x < 1\end{aligned}$$

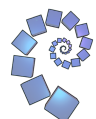
Relationship with sin/cos

$$\begin{aligned}\cos x &= \frac{1}{2}(e^{ix} + e^{-ix}) = \cosh(ix) \\ \cos(iy) &= \frac{1}{2}(e^{i^2y} + e^{-i^2y}) = \cosh(y) \\ \sin x &= \frac{1}{2i}(e^{ix} - e^{-ix}) = \frac{1}{i}\sinh(ix) = -i\sinh(ix) \\ \sin(iy) &= \frac{1}{2i}(e^{i^2y} - e^{-i^2y}) = -\frac{1}{i}\sinh(y) = i\sinh(y)\end{aligned}$$

Inverse hyperbolics¹

$$\begin{aligned}\operatorname{arcosh} x &= \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \quad \text{for } x \geq 1 \\ \operatorname{arsinh} x &= \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \\ \operatorname{artanh} x &= \tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad \text{for } |x| < 1\end{aligned}$$

¹You can find the inverse of $\cosh x$ by setting $\frac{1}{2}(e^y + e^{-y}) = x$ and then solving to get y in terms of x . The other inverse functions can be found in a similar way.



Identities

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

$$\cosh 2x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\operatorname{coth}^2 x - 1 = \operatorname{cosech}^2 x$$

$$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$$

$$\sinh(A \pm B) = \sinh A \cosh B \pm \sinh B \cosh A$$

$$\tanh(A \pm B) = \frac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}$$

Note that the identities for $\sinh 2x$, $\cosh 2x$, $\tanh 2x$ can be derived from the identities for $\sinh(A+B)$ etc.

You can use **Osborn's Rule** to convert trigonometric identities written in terms of sine and cosine into a corresponding hyperbolic identity. Essentially you replace any $\sin^2 \theta$ with $-\sinh^2 \theta$.

Maclaurin's Series

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2r+1}}{(2r+1)!} + \dots \quad \text{for all } x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2r}}{(2r)!} + \dots \quad \text{for all } x$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2r+1}}{(2r+1)} + \dots \quad \text{for } -1 < x < 1$$

Calculus

$$\frac{d}{dx} \sinh x = \cosh x$$

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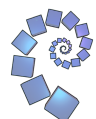
$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$$

$$\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx} \tanh^{-1} x = \frac{1}{1-x^2}$$

Note that you can find $\frac{d}{dx} \operatorname{coth} x$ by using the quotient rule. The chain rule can be used to find the derivatives of $\operatorname{sech} x$ and $\operatorname{cosech} x$.



$$\int \sinh x \, dx = \cosh x + k$$

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$$\int \tanh x \, dx = \ln(\cosh x) + k$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \cosh^{-1}\left(\frac{x}{a}\right) + k \quad \text{or} \quad \ln\left(x + \sqrt{x^2 - a^2}\right) + k \quad \text{for } x > a$$

$$\int \frac{1}{\sqrt{a^2 + x^2}} \, dx = \sinh^{-1}\left(\frac{x}{a}\right) + k \quad \text{or} \quad \ln\left(x + \sqrt{x^2 + a^2}\right) + k$$

$$\int \frac{1}{a^2 - x^2} \, dx = \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right) + k \quad \text{or} \quad \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + k \quad \text{for } |x| < a$$

