STEP Support Programme

STEP 3 Hyperbolic Functions Topic Notes

Definitions

$$\begin{aligned}
\cosh x &= \tfrac{1}{2}\left(\mathrm{e}^{x}+\mathrm{e}^{-x}\right)\\
\sinh x &= \tfrac{1}{2}\left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right)\\
\tanh x &= \frac{\sinh x}{\cosh x} = \frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{\mathrm{e}^{x}+\mathrm{e}^{-x}}\\
\operatorname{sech} x &= \frac{1}{\cosh x}\\
\operatorname{cosech} x &= \frac{1}{\sinh x}\\
\coth x &= \frac{1}{\tanh x} = \frac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{\mathrm{e}^{x}-\mathrm{e}^{-x}}
\end{aligned}$$

Graphs and ranges

The graphs can be sketched by using your understanding of e^x, odd/even functions etc. For example:

$\cosh x$ is an even function (as $\cosh(-x) = \cosh x$) and so it is symmetrical in the y-axis. It passes through $(0, 1)$ and as $x \to \pm\infty$, $y \to \infty$. You can also use Desmos to sketch these.

We have:

$$\begin{aligned}
1 \leqslant \cosh x &< \infty\\
-\infty < \sinh x &< \infty\\
-1 < \tanh x &< 1
\end{aligned}$$

Relationship with sin/cos

$$\begin{aligned}
\cos x &= \tfrac{1}{2}\left(\mathrm{e}^{\mathrm{i}x}+\mathrm{e}^{-\mathrm{i}x}\right) &&= \cosh(\mathrm{i}x)\\
\cos(\mathrm{i}y) &= \tfrac{1}{2}\left(\mathrm{e}^{\mathrm{i}^2y}+\mathrm{e}^{-\mathrm{i}^2y}\right) &&= \cosh(y)\\
\sin x &= \tfrac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{-\mathrm{i}x}\right) &&= \tfrac{1}{\mathrm{i}}\sinh(\mathrm{i}x) = -\mathrm{i}\sinh(\mathrm{i}x)\\
\sin(\mathrm{i}y) &= \tfrac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}^2y}-\mathrm{e}^{-\mathrm{i}^2y}\right) &&= -\tfrac{1}{\mathrm{i}}\sinh(y) = \mathrm{i}\sinh(y)
\end{aligned}$$

Inverse hyperbolics[1]

$$\begin{aligned}
\operatorname{arcosh} x &= \cosh^{-1} x = \ln\left(x+\sqrt{x^2-1}\right) && \text{for } x \geqslant 1\\
\operatorname{arsinh} x &= \sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)\\
\operatorname{artanh} x &= \tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) && \text{for } |x| < 1
\end{aligned}$$

[1] You can find the inverse of $\cosh x$ by setting $\frac{1}{2}\left(\mathrm{e}^{y}+\mathrm{e}^{-y}\right) = x$ and then solving to get y in terms of x. The other inverse functions can be found in a similar way.

Identities

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2\sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$$
$$\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$
$$1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\coth^2 x - 1 = \operatorname{cosech}^2 x$$
$$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$$
$$\sinh(A \pm B) = \sinh A \cosh B \pm \sinh B \cosh A$$
$$\tanh(A \pm B) = \frac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}$$

Note that the identities for $\sinh 2x$, $\cosh 2x$, $\tanh 2x$ can be derived from the identities for $\sinh(A+B)$ etc.

You can use Osborn's Rule to covert trigonometric identities written in terms of sine and cosine into a corresponding hyperbolic identity. Essentially you replace any $\sin^2\theta$ with $-\sinh^2\theta$.

Maclaurin's Series

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{for all } x$$
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots \quad \text{for all } x$$
$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{(2r+1)} + \ldots \quad \text{for } -1 < x < 1$$

Calculus

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh x = \operatorname{sech}^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1} x = \frac{1}{1-x^2}$$

Note that you can find $\frac{\mathrm{d}}{\mathrm{d}x}\coth x$ by using the quotient rule. The chain rule can be used to find the derivatives of $\operatorname{sech} x$ and $\operatorname{cosech} x$.

$$\int \sinh x \,\mathrm{d}x = \cosh x + k$$

$$\int \cosh x \,\mathrm{d}x = \sinh x + k$$

$$\int \tanh x \,\mathrm{d}x = \ln(\cosh x) + k$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \,\mathrm{d}x = \cosh^{-1}\left(\frac{x}{a}\right) + k \quad \text{or} \quad \ln\left(x + \sqrt{x^2 - a^2}\right) + k \quad \text{for } x > a$$

$$\int \frac{1}{\sqrt{a^2 + x^2}} \,\mathrm{d}x = \sinh^{-1}\left(\frac{x}{a}\right) + k \quad \text{or} \quad \ln\left(x + \sqrt{x^2 + a^2}\right) + k$$

$$\int \frac{1}{a^2 - x^2} \,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + k \quad \text{or} \quad \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| + k \quad \text{for } |x| < a$$