

STEP Support Programme

STEP 2 Matrices Questions

This collection of questions is different from most of the STEP Support Programme, since matrices have not been on the STEP syllabus for many years. Note the following:

- These are **not** past STEP questions; they are from old A-level papers and similar.
- Many of these questions are *longer* or *shorter* than a typical STEP question.
- Many of these questions are *easier* or *harder* than a typical STEP question.
- The questions appear in chronological order of their origin, **not** in approximate order of difficulty.

The questions have been chosen to challenge you to think about matrices in a more sophisticated way than current A-level questions are likely to, and so will be a good preparation for what might appear on future STEP papers.

Matrices will first be examinable on STEP papers 2 and 3 from 2019 (under the new specification). There were a small number of STEP questions on the topic of matrices in the 1980s and 1990s. These can be found by searching for ‘matrices’ on the [STEP database](#). Some of these are appropriate for today’s STEP paper 2 or 3, but others require content beyond the current specification.

Acknowledgements (copyright details and question sources) for the questions in this module appear on the final page.



- 1 The complex number $x + iy$ is mapped into the complex number $X + iY$ where X and Y are given by the equation

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Which numbers are invariant under the mapping?

- 2 The simultaneous equations

$$x + 2y = 4,$$

$$2x - y = 0,$$

$$3x + y = 5$$

may be written in matrix form as

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}, \quad \text{or} \quad \mathbf{AX} = \mathbf{B}.$$

Carry out numerically the procedure of the three following steps:

- (1) $\mathbf{A}^T \mathbf{AX} = \mathbf{A}^T \mathbf{B}$;
- (2) $(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{AX} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{B}$;
- (3) $\mathbf{IX} = \begin{pmatrix} x \\ y \end{pmatrix} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{B}$.

Verify that the values of x, y so found do not satisfy all the original three equations. Suggest a reason for this.

Under what circumstances will the procedure given above, when applied to a set of three simultaneous equations in two variables, result in values which satisfy the equations?

Note: The final part of this question is challenging to answer fully; a complete solution is beyond what would be expected on a STEP examination.



3 Let \mathbf{A} , \mathbf{B} , \mathbf{C} be real 2×2 matrices and write

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}, \text{ etc.}$$

Prove that:

- (i) $[\mathbf{A}, \mathbf{A}] = \mathbf{O}$, where \mathbf{O} is the zero matrix,
- (ii) $[[\mathbf{A}, \mathbf{B}], \mathbf{C}] + [[\mathbf{B}, \mathbf{C}], \mathbf{A}] + [[\mathbf{C}, \mathbf{A}], \mathbf{B}] = \mathbf{O}$,
- (iii) if $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$, then $[\mathbf{A}, \mathbf{B}^m] = m\mathbf{B}^{m-1}$ for all positive integers m .

At each step you should state clearly any properties of matrices which you use.

The *trace*, $\text{Tr}(\mathbf{A})$, of a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is defined by

$$\text{Tr}(\mathbf{A}) = a_{11} + a_{22}.$$

Prove that:

- (iv) $\text{Tr}(\mathbf{A} + \mathbf{B}) = \text{Tr}(\mathbf{A}) + \text{Tr}(\mathbf{B})$,
- (v) $\text{Tr}(\mathbf{AB}) = \text{Tr}(\mathbf{BA})$,
- (vi) $\text{Tr}(\mathbf{I}) = 2$.

Deduce that there are no matrices satisfying $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$. Does this in any way invalidate the statement in (iii)?

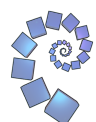
4 Matrices \mathbf{P} and \mathbf{Q} are given by

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

(where $i^2 = -1$). Show that $\mathbf{P}^2 = \mathbf{Q}^2$, $\mathbf{PQP} = \mathbf{Q}$ and $\mathbf{P}^4 = \mathbf{I}$, the identity matrix. Deduce that, for all positive integers n , $\mathbf{P}^n \mathbf{Q} \mathbf{P}^n = \mathbf{Q}$. Hence, or otherwise, show that if \mathbf{X} and \mathbf{Y} are each matrices of the form

$$\mathbf{P}^m \mathbf{Q}^n, \quad m = 1, 2, 3, 4; \quad n = 1, 2$$

then \mathbf{XY} has the same form.



- 5 (a) Show that if $\mathbf{A} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, then

$$\mathbf{A}^2 - (p + s)\mathbf{A} + (ps - qr)\mathbf{I} = \mathbf{O},$$

where \mathbf{I} is the identity matrix and \mathbf{O} is the zero matrix.

- (b) Given that $\mathbf{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and that $\mathbf{X}^2 = \mathbf{O}$, show that \mathbf{X} can be written either in terms of a and b only or in terms of c only, or of b only.

Show that when \mathbf{X} is written in terms of c only, the solution can be written in the form:

$$\mathbf{X} = c \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and interpret this result in terms of transformations of the plane represented by these matrices, relating your answer to the fact that $\mathbf{X}^2 = \mathbf{O}$.

- 6 A mapping $(x, y) \rightarrow (u, v)$ is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -8 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Show briefly that this mapping is not one to one.

Find the locus, L , of all points which map to $(1, -4)$. Describe the locus of (u, v) as (x, y) is allowed to vary throughout the plane. Show that any given point, P , on this locus is the image of just one point on the y -axis, and describe how the set of all points with image P is related to the locus L .

- 7 You are given that \mathbf{P} , \mathbf{Q} and \mathbf{R} are 2×2 matrices, \mathbf{I} is the identity matrix and \mathbf{P}^{-1} exists.

- (i) Prove, by expanding both sides, that

$$\det(\mathbf{PQ}) = \det \mathbf{P} \det \mathbf{Q}.$$

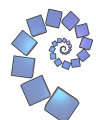
Deduce that

$$\det(\mathbf{P}^{-1}\mathbf{Q} + \mathbf{I}) = \det(\mathbf{QP}^{-1} + \mathbf{I}).$$

- (ii) If $\mathbf{PX} = \mathbf{XP}$ for every 2×2 matrix \mathbf{X} , prove that $\mathbf{P} = \lambda\mathbf{I}$, where λ is a constant.
 (iii) If $\mathbf{RQ} = \mathbf{QR}$, prove that

$$\mathbf{RQ}^n = \mathbf{Q}^n\mathbf{R} \quad \text{and} \quad \mathbf{R}^n\mathbf{Q}^n = \mathbf{Q}^n\mathbf{R}^n$$

for any positive integer n .

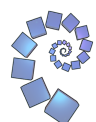


8 The real 3×3 matrix \mathbf{A} is such that $\mathbf{A}^2 = \mathbf{A}$.

- (i) Prove that $(\mathbf{I} - \mathbf{A})^2 = \mathbf{I} - \mathbf{A}$.
- (ii) Express $(\mathbf{I} - \mathbf{A})^3$ in the form $\mathbf{I} + k\mathbf{A}$, where k is a number to be determined.
- (iii) Prove that, for all real constants λ and all positive integers n ,

$$(\mathbf{I} + \lambda\mathbf{A})^n = \mathbf{I} + ((\lambda + 1)^n - 1)\mathbf{A}.$$

Use this result to verify your answer to (ii).



Acknowledgements

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In the list of sources below, the following abbreviations are used:

O&C	Oxford and Cambridge Schools Examination Board
SMP	School Mathematics Project
MEI	Mathematics in Education and Industry
QP	Question paper
Q	Question

- 1 O&C, A level Mathematics (SMP), 1966, QP Mathematics II, Q A3
- 2 O&C, A level Mathematics (SMP), 1967, QP Mathematics II, Q B22
- 3 O&C, A level Mathematics (MEI), 1968, QP MEI 20, Pure Mathematics III (Special Paper), Q 3; editorial changes here: the definition of \mathbf{O} is inserted, the implication symbol is written in words, and the reference to the matrix ring is removed
- 4 O&C, A level Mathematics (MEI), 1968, QP MEI 143*, Pure Mathematics I, Q 6
- 5 O&C, A level Mathematics (MEI), 1980, QP 9655/1, Pure Mathematics 1, Q 2; editorial changes here: use \mathbf{O} rather than $\mathbf{0}$ for the zero matrix, and define the notation.
- 6 O&C, A level Mathematics (MEI), 1981, QP 9655/1, Pure Mathematics 1, Q 6(b)
- 7 O&C, A level Mathematics (MEI), 1986, QP 9657/0, Mathematics 0 (Special Paper), Q 2; editorial change here: \mathbf{I} is called the identity matrix rather than the unit matrix
- 8 O&C, A level Mathematics (MEI), 1987, QP 9650/2, Mathematics 2, Q 16

