STEP Support Programme

STEP 2 Matrices Questions

This collection of questions is different from most of the STEP Support Programme, since matrices have not been on the STEP syllabus for many years. Note the following:

- These are **not** past STEP questions; they are from old A-level papers and similar.
- Many of these questions are *longer* or *shorter* than a typical STEP question.
- Many of these questions are *easier* or *harder* than a typical STEP question.
- The questions appear in chronological order of their origin, **not** in approximate order of difficulty.

The questions have been chosen to challenge you to think about matrices in a more sophisticated way than current A-level questions are likely to, and so will be a good preparation for what might appear on future STEP papers.

Matrices will first be examinable on STEP papers 2 and 3 from 2019 (under the new specification). There were a small number of STEP questions on the topic of matrices in the 1980s and 1990s. These can be found by searching for 'matrices' on the STEP database. Some of these are appropriate for today's STEP paper 2 or 3, but others require content beyond the current specification.

Acknowledgements (copyright details and question sources) for the questions in this module appear on the final page.

1 The complex number $x + \mathrm{i}y$ is mapped into the complex number $X + \mathrm{i}Y$ where X and Y are given by the equation

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Which numbers are invariant under the mapping?

2 The simultaneous equations

$$\begin{aligned} x + 2y &= 4, \\ 2x - y &= 0, \\ 3x + y &= 5 \end{aligned}$$

may be written in matrix form as

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}, \quad \text{or} \quad \mathbf{AX} = \mathbf{B}.$$

Carry out numerically the procedure of the three following steps:

(1) $\mathbf{A}^{\mathrm{T}}\mathbf{AX} = \mathbf{A}^{\mathrm{T}}\mathbf{B}$;

(2) $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{AX} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{B}$;

(3) $\mathbf{IX} = \begin{pmatrix} x \\ y \end{pmatrix} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{B}$.

Verify that the values of x, y so found do not satisfy all the original three equations. Suggest a reason for this.

Under what circumstances will the procedure given above, when applied to a set of three simultaneous equations in two variables, result in values which satisfy the equations?

Note: The final part of this question is challenging to answer fully; a complete solution is beyond what would be expected on a STEP examination.

3 Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be real 2×2 matrices and write

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}, \text{ etc.}$$

Prove that:

(i) $[\mathbf{A}, \mathbf{A}] = \mathbf{O}$, where $\mathbf{O}$ is the zero matrix,

(ii) $[[\mathbf{A}, \mathbf{B}], \mathbf{C}] + [[\mathbf{B}, \mathbf{C}], \mathbf{A}] + [[\mathbf{C}, \mathbf{A}], \mathbf{B}] = \mathbf{O}$,

(iii) if $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$, then $[\mathbf{A}, \mathbf{B}^m] = m\mathbf{B}^{m-1}$ for all positive integers m.

At each step you should state clearly any properties of matrices which you use.

The *trace*, $\text{Tr}(\mathbf{A})$, of a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is defined by

$$\text{Tr}(\mathbf{A}) = a_{11} + a_{22}.$$

Prove that:

(iv) $\text{Tr}(\mathbf{A} + \mathbf{B}) = \text{Tr}(\mathbf{A}) + \text{Tr}(\mathbf{B})$,

(v) $\text{Tr}(\mathbf{AB}) = \text{Tr}(\mathbf{BA})$,

(vi) $\text{Tr}(\mathbf{I}) = 2$.

Deduce that there are no matrices satisfying $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$. Does this in any way invalidate the statement in (iii)?

4 Matrices $\mathbf{P}$ and $\mathbf{Q}$ are given by

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$$

(where $\mathrm{i}^2 = -1$). Show that $\mathbf{P}^2 = \mathbf{Q}^2$, $\mathbf{PQP} = \mathbf{Q}$ and $\mathbf{P}^4 = \mathbf{I}$, the identity matrix. Deduce that, for all positive integers n, $\mathbf{P}^n\mathbf{Q}\mathbf{P}^n = \mathbf{Q}$. Hence, or otherwise, show that if $\mathbf{X}$ and $\mathbf{Y}$ are each matrices of the form

$$\mathbf{P}^m\mathbf{Q}^n, \quad m = 1, 2, 3, 4; \ n = 1, 2$$

then $\mathbf{XY}$ has the same form.

5 (a) Show that if $\mathbf{A} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, then

$$\mathbf{A}^2 - (p+s)\mathbf{A} + (ps - qr)\mathbf{I} = \mathbf{O},$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{O}$ is the zero matrix.

(b) Given that $\mathbf{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and that $\mathbf{X}^2 = \mathbf{O}$, show that $\mathbf{X}$ can be written either in terms of a and b only or in terms of c only, or of b only.

Show that when $\mathbf{X}$ is written in terms of c only, the solution can be written in the form:

$$\mathbf{X} = c \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and interpret this result in terms of transformations of the plane represented by these matrices, relating your answer to the fact that $\mathbf{X}^2 = \mathbf{O}$.

6 A mapping $(x, y) \to (u, v)$ is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -8 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Show briefly that this mapping is not one to one.

Find the locus, L, of all points which map to $(1, -4)$. Describe the locus of (u, v) as (x, y) is allowed to vary throughout the plane. Show that any given point, P, on this locus is the image of just one point on the y-axis, and describe how the set of all points with image P is related to the locus L.

7 You are given that $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ are 2×2 matrices, $\mathbf{I}$ is the identity matrix and $\mathbf{P}^{-1}$ exists.

(i) Prove, by expanding both sides, that

$$\det(\mathbf{PQ}) = \det \mathbf{P} \det \mathbf{Q}.$$

Deduce that

$$\det(\mathbf{P}^{-1}\mathbf{Q} + \mathbf{I}) = \det(\mathbf{Q}\mathbf{P}^{-1} + \mathbf{I}).$$

(ii) If $\mathbf{PX} = \mathbf{XP}$ for *every* 2×2 matrix $\mathbf{X}$, prove that $\mathbf{P} = \lambda\mathbf{I}$, where λ is a constant.

(iii) If $\mathbf{RQ} = \mathbf{QR}$, prove that

$$\mathbf{R}\mathbf{Q}^n = \mathbf{Q}^n\mathbf{R} \quad \text{and} \quad \mathbf{R}^n\mathbf{Q}^n = \mathbf{Q}^n\mathbf{R}^n$$

for any positive integer n.

8 The real 3×3 matrix $\mathbf{A}$ is such that $\mathbf{A}^2 = \mathbf{A}$.

(i) Prove that $(\mathbf{I} - \mathbf{A})^2 = \mathbf{I} - \mathbf{A}$.

(ii) Express $(\mathbf{I} - \mathbf{A})^3$ in the form $\mathbf{I} + k\mathbf{A}$, where k is a number to be determined.

(iii) Prove that, for all real constants λ and all positive integers n,

$$(\mathbf{I} + \lambda\mathbf{A})^n = \mathbf{I} + \big((\lambda + 1)^n - 1\big)\mathbf{A}.$$

Use this result to verify your answer to (ii).

Acknowledgements

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In the list of sources below, the following abbreviations are used:

O&C	Oxford and Cambridge Schools Examination Board
SMP	School Mathematics Project
MEI	Mathematics in Education and Industry
QP	Question paper
Q	Question

1 O&C, A level Mathematics (SMP), 1966, QP Mathematics II, Q A3

2 O&C, A level Mathematics (SMP), 1967, QP Mathematics II, Q B22

3 O&C, A level Mathematics (MEI), 1968, QP MEI 20, Pure Mathematics III (Special Paper), Q 3; editorial changes here: the definition of **O** is inserted, the implication symbol is written in words, and the reference to the matrix ring is removed

4 O&C, A level Mathematics (MEI), 1968, QP MEI 143*, Pure Mathematics I, Q 6

5 O&C, A level Mathematics (MEI), 1980, QP 9655/1, Pure Mathematics 1, Q 2; editorial changes here: use **O** rather than **0** for the zero matrix, and define the notation.

6 O&C, A level Mathematics (MEI), 1981, QP 9655/1, Pure Mathematics 1, Q 6(b)

7 O&C, A level Mathematics (MEI), 1986, QP 9657/0, Mathematics 0 (Special Paper), Q 2; editorial change here: **I** is called the identity matrix rather than the unit matrix

8 O&C, A level Mathematics (MEI), 1987, QP 9650/2, Mathematics 2, Q 16