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STEP Support Programme

STEP 3 Matrices Hints

Question 1

Express the determinant

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

as the product of factors which are linear in a, b, c.

This determinant has a lot of symmetry, so using row or column operations (see the STEP 3 Matrix Topic Notes for more on this) and then looking for common factors in the rows or columns is more likely to lead to an efficient solution than first expanding the determinant and then hoping to find a factorisation.

Another approach is to make use of the factor theorem: we know from this theorem that $a - k$ is a factor of the determinant if setting $a = k$ makes the determinant zero.

Hence, or otherwise, find $x : y : z : u$ if

$$\begin{aligned} x + 2\ y + 3\ z + 4\ u &= 0, \\ x + 2^2 y + 3^2 z + 4^2 u &= 0, \\ x + 2^3 y + 3^3 z + 4^3 u &= 0. \end{aligned}$$

This could be solved directly, but the similarity to the previous part of the question suggests making use of it to write down the determinant of the matrix.

Question 2

Prove that $(a-b)$ and $(x-y)$ are factors of the determinant

$$\begin{vmatrix} (a+x)^2 & (a+y)^2 & (a+z)^2 \\ (b+x)^2 & (b+y)^2 & (b+z)^2 \\ (c+x)^2 & (c+y)^2 & (c+z)^2 \end{vmatrix}$$

and factorise the determinant completely.

The use of the word "factor" in the question suggests making use of the factor theorem. For example, we know from this theorem that $a-b$ is a factor of the determinant if and only if setting $a=b$ makes the determinant zero.

As this determinant has a lot of symmetry, another approach is to use row or column operations (see the STEP 3 Matrix Topic Notes for more on this), and then look for common factors in the rows or columns in the simplified determinant. This is more likely to lead to a nicely factorised expressed much more efficiently than first expanding the determinant and then hoping to find a factorisation.

Question 3

Given that α, β, γ are the roots of the equation

$$\begin{vmatrix} a & b & x \\ x & c & a \\ c & x & b \end{vmatrix} = 0,$$

prove that

$$\alpha^3 + \beta^3 + \gamma^3 = -6abc.$$

Consider what you know about manipulating roots of polynomials. $\alpha^3 + \beta^3 + \gamma^3$ is a symmetric polynomial in the roots of the (cubic) equation given by the determinant, so it can be expressed in terms of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$.

Another perspective is to note that since α is a root of the polynomial, it satisfies it, and likewise for β and γ. How might this be useful?

Question 4

Prove that if the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}\right\}$$

are simultaneously satisfied by values of x, y, z which are not all zero, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

How can we use matrices to solve the three simultaneous equations? What is the "obvious" solution to the equations? In what situation could we get a solution with not all of x, y, z being zero?

Hence, or otherwise, eliminate x, y, z from the equations

$$a = \frac{x}{y-z}, \quad b = \frac{y}{z-x}, \quad c = \frac{z}{x-y}.$$

[Your answer should **not** be left in determinant form.]

In the current form of these equations, we have variables in the denominators of fractions. How can you rearrange things so that no fractions are left?

Once you have linear equations in x, y and z, how can you use the previous part to find some relation between a, b and c? What property do x, y and z have that allows you to do this?

Question 5

A matrix $\mathbf{M}$ is said to be transposed into the matrix $\mathbf{M}^{\mathrm{T}}$ if the first row of $\mathbf{M}$ becomes the first column of $\mathbf{M}^{\mathrm{T}}$, the second row of $\mathbf{M}$ becomes the second column of $\mathbf{M}^{\mathrm{T}}$, and so on. Write down the transposes of the matrices

$$\mathbf{M} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{T} = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & c \\ a & 0 & 0 \end{pmatrix}.$$

Calculate the matrix products $\mathbf{M}^{\mathrm{T}}\mathbf{M}$ and $\mathbf{TM}$; show also that $(\mathbf{TM})^{\mathrm{T}} = \mathbf{M}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}$.

This is straightforward calculation. Note that $\mathbf{M}^{\mathrm{T}}\mathbf{M}$ is a 1×1 matrix.

If the elements of $\mathbf{M}$ are the Cartesian coordinates of a point P, what information is provided by the element of $\mathbf{M}^{\mathrm{T}}\mathbf{M}$?

The element should look like a formula you are familiar with.

If the matrix $\mathbf{T}$ describes a transformation of the points P of three dimensional space, interpret geometrically the equation

$$(\mathbf{TM})^{\mathrm{T}}(\mathbf{TM}) = \mathbf{M}^{\mathrm{T}}\mathbf{M},$$

and find all appropriate values of a, b and c for which this equation holds for all points P.

Note that $\mathbf{TM}$ is the position vector of the image of P under the transformation described by the matrix $\mathbf{T}$. Use the previous part to interpret the meaning of $(\mathbf{TM})^{\mathrm{T}}(\mathbf{TM})$, and write out the given equation in terms of x, y and so on.

If the equation is to be true for all points P, then you can help yourself by choosing really simple points to work with.

Note that once you have shown that *if* the equation holds for all points P, then a has to have a certain value or one of a certain set of values, and so on, you still have to show that *if* a has this value or one of these values, and so on, *then* the equation holds.

Question 6

The functions $t \to \mathbf{P}$, $t \to \mathbf{Q}$ map real numbers t onto matrices $\mathbf{P}$, $\mathbf{Q}$ of fixed dimension; that is, each element of each matrix is a real function of t, and we suppose all the functions to be differentiable. A scalar multiplier s is also a function of t. The derivative of a matrix is defined as a matrix whose elements are the derivatives of the elements of the original matrix. We write $\mathrm{d}\mathbf{P}/\mathrm{d}t$ as $\dot{\mathbf{P}}$, $\mathrm{d}s/\mathrm{d}t$ as $\dot{s}$, and so on.

Before jumping into proving the results requested, it would be helpful to write down an expression for $\dot{\mathbf{P}}$ itself.

(i) Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}(s\mathbf{P}) = \dot{s}\mathbf{P} + s\dot{\mathbf{P}}.$$

Can you write down the left-hand side explicitly? What about the right-hand side? Can you show that these are equal using the ordinary product rule?

(ii) If the product $\mathbf{PQ}$ is defined, prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{PQ}) = \dot{\mathbf{P}}\mathbf{Q} + \mathbf{P}\dot{\mathbf{Q}}.$$

The challenge here is to not get lost in a sea of symbols. Can you find a clear way of writing the (i, j)th element of the left-hand side and of the right-hand side?

(iii) Prove that the derivative of a constant matrix is the zero matrix.

A constant matrix is one where every element is a constant, that is, no element depends on t.

(iv) If $\mathbf{M}$ is the rotation matrix

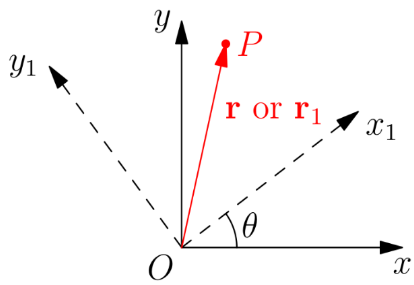
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

θ being a function of t, prove that $\dot{\mathbf{M}} = \mathbf{MJ}\dot{\theta}$, where

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This does not require the use of any of the earlier parts. Can you calculate the two sides of the equation and show that they are equal?

The position of a particle in a plane is specified by a vector which can be described either as $\mathbf{r}$ relative to a coordinate system with rectangular axes Ox, Oy, or as $\mathbf{r}_1$ relative to a coordinate system with axes Ox_1, Oy_1, as shown in the diagram. The angle xOx_1, denoted by θ, is a function of t.



Write an equation connecting $\mathbf{r}$ with $\mathbf{r}_1$, and prove that

$$\dot{\mathbf{r}} = \mathbf{M}(\dot{\mathbf{r}}_1 + \mathbf{J}\mathbf{r}_1\dot{\theta}).$$

Can you express $\mathbf{r}$ in terms of coordinates? What about $\mathbf{r}_1$? How are the coordinates of these two vectors related? Can you write $\mathbf{r} = \mathbf{A}\mathbf{r}_1$ or $\mathbf{r}_1 = \mathbf{A}\mathbf{r}$ for some matrix $\mathbf{A}$?

You could either work in terms of the Cartesian coordinates of P or in terms of the polar coordinates.

Once you have this, then you can differentiate this equation to prove the requested result, making use of the earlier parts of the question.

Question 7

If

$$x = \mathrm{e}^{kt}(a\cos\lambda t + b\sin\lambda t)$$

show that

$$\dot{x} = \mathrm{e}^{kt}(a'\cos\lambda t + b'\sin\lambda t),$$

where

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} k & \lambda \\ -\lambda & k \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and dot denotes differentiation with respect to t, and find an expression for $\ddot{x}$ with coefficients given in a similar way.

Simply perform the differentiation to find $\dot{x}$ and differentiate again to obtain $\ddot{x}$.

Alternatively, since $\dot{x}$ is exactly the same as x, just with a and b replaced by a' and b', you could use your result for $\dot{x}$ to find $\ddot{x}$ without a second explicit differentiation.

A particular integral solution to

$$\ddot{x} + 2p\dot{x} + q^2x = \mathrm{e}^{kt}(C\cos\lambda t + D\sin\lambda t)$$

is

$$x = \mathrm{e}^{kt}(a\cos\lambda t + b\sin\lambda t).$$

Show that

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} (k^2-\lambda^2) + 2pk + q^2 & 2k\lambda + 2p\lambda \\ -2k\lambda - 2p\lambda & (k^2-\lambda^2) + 2pk + q^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and hence write $\begin{pmatrix} a \\ b \end{pmatrix}$ in terms of $\begin{pmatrix} C \\ D \end{pmatrix}$.

Discuss any particular cases.

You now have expressions for $\ddot{x}$ and $\dot{x}$ for the given x; what happens when you substitute these into the differential equation?

Once you have $\begin{pmatrix} C \\ D \end{pmatrix} = \mathbf{A}\begin{pmatrix} a \\ b \end{pmatrix}$, where $\mathbf{A}$ is a matrix, how can you rearrange this to get $\begin{pmatrix} a \\ b \end{pmatrix} = \cdots$?

For the question about particular cases, what assumption or assumptions have you made when writing $\begin{pmatrix} a \\ b \end{pmatrix}$ in terms of $\begin{pmatrix} C \\ D \end{pmatrix}$? What happens if these assumptions do not hold?

Question 8

Sketch the graph whose equation is

$$x^2 - y^2 = a^2 \quad (a > 0).$$

You may be familiar with this equation. If you are not, you can consider aspects of the graph such as:

- Where does the graph intersect the axes?
- What happens to y as $x \to \pm\infty$?
- What happens to x as $y \to \pm\infty$?
- Does the graph have any stationary points? (This is a little harder as the equation is given implicitly.)
- Does the graph have any asymptotes?

Prove that the point $P(a\cosh t, a\sinh t)$ lies on this graph for all real values of t; but that there are points of the graph which cannot be expressed in this form.

To show that P lies on the graph, you need only show that the coordinates of P satisfy the equation.

Looking at your sketch of the graph and considering the possible values of $a\cosh t$ and $a\sinh t$ should convince you that there are points on the graph which cannot be expressed in this form.

If U is the point of the graph for which $t = u$, find the area of the region $\mathscr{R}$ bounded by the curve, the line OU and the x-axis. (You may assume that $u > 0$.)

Use your sketch to work out what the region looks like! That will suggest a strategy for calculating the area.

You will clearly need to use the formula for working out the area under a curve when it is described parametrically.

Prove that the transformation with matrix

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \qquad (\alpha \neq 0)$$

transforms the point P into another point of the curve. Into what region is $\mathscr{R}$ transformed by this? What is the area of the transformed region? Give reasons for your answers.

You could work out where P is transformed to. You will also need to work out how the boundaries of the region $\mathscr{R}$ are transformed.

Remember that a transformation represented by a matrix is a *linear transformation*: it transforms straight lines into straight lines, and it fixes the origin.

Sketch the resulting region!

You could use the sketch to work out the area of the transformed region, or you could use what you know about the area-transforming effects of a matrix.

Question 9

Show that the equations

$$\begin{aligned} 3x + 2y + z &= a - 1, \\ -2x + (a-2)y - az &= 2a, \\ 6x + ay + (a-2)z &= 3a - 6 \end{aligned}$$

have a solution, not necessarily unique, unless $a = \frac{2}{3}$.

Find the complete solution when $a = 0$ and when $a = 4$.

The first part of the question could be rewritten as: show that when $a \neq \frac{2}{3}$, the equations have a solution (which is not necessarily unique), and that when $a = \frac{2}{3}$, the equations do not have any solutions. So you have to do both of these things.

You could try to solve these equations directly, or you could write them as a matrix equation.

Question 10

If z is the complex number $x + \mathrm{i}y$, $\mathrm{i} = \sqrt{-1}$, let $\mathbf{M}(z)$ denote the 2×2 matrix

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

Prove that

$$\mathbf{M}(z + z') = \mathbf{M}(z) + \mathbf{M}(z')$$

and that

$$\mathbf{M}(zz') = \mathbf{M}(z)\mathbf{M}(z').$$

For the first part, you will need to work out $\mathbf{M}(z + z')$ and $\mathbf{M}(z) + \mathbf{M}(z')$ separately, using the definition of $\mathbf{M}(z)$, and show that they are equal. The second part is similar.

Hence, or otherwise, show that

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}.$$

The complex number represented by the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is .... This should look familiar! What happens when you raise it to the power of n?

Hence find three real 2×2 matrices $\mathbf{A}$ such that $\mathbf{A}^3 = \mathbf{I}$ and a 2×2 matrix $\mathbf{B}$ such that $\mathbf{B}^2 + \mathbf{I} = \mathbf{B}$.

If this question is all about complex numbers being represented by matrices, then what could you do for these equations?

Question 11

Using any method you wish, calculate the inverse $\mathbf{A}^{-1}$ of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 & 4 & 6 \end{pmatrix}.$$

This is routine; you can use any method you are comfortable with.

Interpret geometrically in three dimensions the equations

$$\begin{aligned} x + y + 2z &= 4, \\ 2x + 3y + 3z &= 8, \\ 3x + 4y + \lambda z &= 7 + \lambda \end{aligned}$$

in the cases $\lambda = 6$ and $\lambda = 5$.

The case $\lambda = 6$ clearly corresponds to the above matrix. Annoyingly, $\lambda = 5$ requires you to start again (unless you replaced 6 by λ in the first part!).

Writing out the three equations and staring at them for a bit might help here! The examiners are probably not expecting you to repeat all of the work you did on the previous part.

Note also that the question asks for a geometrical interpretation. Do you need to solve the equations or attempt to do so to interpret the equations geometrically?

Question 12

Using any method you prefer, and solving the problems in any order you prefer, find

(i) the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & 7 \\ 6 & 8 & 3 \end{pmatrix},$$

(ii) the solution of the simultaneous equations

$$\begin{aligned} x + 2y + 4z &= 3, \\ 3x + 5y + 7z &= 12, \\ 6x + 8y + 3z &= 31. \end{aligned}$$

It is clear that the simultaneous equations can be written using the matrix in part (i). So it probably makes sense to find the inverse of the matrix first.

The rest of this part of the question is routine calculation.

The last of the three equations is replaced by

$$6x + 8y + az = b,$$

and it is found that the first two equations together with the new third one have more than one solution. Find a and b, and state a geometrical interpretation in three dimensions for these equations.

You might find it useful to review the STEP 3 Matrices Topic Notes, where the geometric interpretation of three simultaneous equations in three unknowns is discussed in detail.

Once you have decided what the geometry must be, it will be relatively straightforward to work out the equation of this third plane.

Question 13

$\mathbf{A}$ is a 3×3 matrix whose elements are 0, 1 or -1 and each row and each column of $\mathbf{A}$ contains exactly one non-zero element. Prove that $\mathbf{A}^2$, $\mathbf{A}^3$, $\ldots$, $\mathbf{A}^n$ are all of the same form and deduce that $\mathbf{A}^h = \mathbf{I}$ for some positive integer $h \leqslant 48$.

The first result asked for (that the powers of $\mathbf{A}$ all have this form) is relatively easy to see by trying an example or two, but is quite fiddly to prove rigorously.

Since the result is talking about $\mathbf{A}^n$ for any positive integer n, induction is clearly an appropriate approach.

For the second part, consider how many possible matrices of the given form there are. What would happen if the result were false, that is if $\mathbf{A}^h \neq \mathbf{I}$ for any positive integer $h \leqslant 48$?

Interpret the action of $\mathbf{A}$ on a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ geometrically.

This is a vague question! It depends on the precise matrix $\mathbf{A}$. But can you break down the possible matrices of this form into simpler transformations, and describe the action as a composition of some number of simple transformations?

Question 14

Find all possible solutions to

$$\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ k \end{pmatrix} \qquad \text{(i)},$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{pmatrix},$$

stating explicitly the value of k that gives these solutions.

Were the k a fixed number, this would be a standard question. As it is, the question means: There is some value of k for which the equation has a solution. For this value of k, find all possible solutions.

The second part of the question suggests a plausible value for k.

The equations

$$\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

may each be regarded as the equations of three planes; give a geometrical interpretation of your solution of equation (i) in terms of these sets of planes.

One of these will most likely correspond to the value of k found earlier; what do the three planes look like in this case? In the other case, what has happened to the planes?

Write down the equation of the line, L, of intersection of the planes

$$x + 2y - z = 2 \quad \text{and} \quad 2x + y + z = 1.$$

This should be immediate from the earlier parts of the question.

Find the equation of the plane through $(0, 1, 0)$ perpendicular to L and the coordinates of the points where this plane intersects the lines of intersection of

$$x + 2y - z = 2 \quad \text{and} \quad 5x + 4y + z = 3$$

and

$$2x + y + z = 1 \quad \text{and} \quad 5x + 4y + z = 3.$$

Though this part of the question (which is all about vectors, lines and planes) follows on from the earlier parts of the question, it is not clear how we can use the earlier results to help us.

It therefore makes sense to fall back on what you know about vectors instead.

Question 15

(i) Given $\mathbf{M} = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}$, calculate $\mathbf{M}^2$ and $\mathbf{M}^3$.

Suggest a form for $\mathbf{M}^n$ and confirm your suggestion, using the method of proof by induction.

Once you have a suggestion for $\mathbf{M}^n$, do check that it also works in the case $n = 1$! Then you can prove your result using induction, as instructed.

(ii) Prove that, for any $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{AB} = \mathbf{BA}$$

if and only if $(\mathbf{A} - k\mathbf{I})(\mathbf{B} - k\mathbf{I}) = (\mathbf{B} - k\mathbf{I})(\mathbf{A} - k\mathbf{I})$ for all values of the real number k.

Note that this question asks us to prove that something is true *if and only if* something else is true. So we have to prove both directions, that is,

- *if* $\mathbf{AB} = \mathbf{BA}$, *then* $(\mathbf{A} - k\mathbf{I})(\mathbf{B} - k\mathbf{I}) = (\mathbf{B} - k\mathbf{I})(\mathbf{A} - k\mathbf{I})$ for all values of the real number k, and
- *if* $(\mathbf{A} - k\mathbf{I})(\mathbf{B} - k\mathbf{I}) = (\mathbf{B} - k\mathbf{I})(\mathbf{A} - k\mathbf{I})$ for all values of the real number k, *then* $\mathbf{AB} = \mathbf{BA}$.

Expanding the brackets should help here.

(iii) Prove that, for any $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$, where $\mathbf{A}^{\mathrm{T}}$ is the transpose of $\mathbf{A}$.

The most difficult part of this problem is to find a way of expressing the elements of the matrix $\mathbf{AB}$ in an algebraic form. Remember that we can write

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

so the (i, j)th element of $\mathbf{A}$ is a_{ij}.

Writing $\mathbf{B}$ in a similar way, and thinking carefully about how we multiply two matrices or find the transpose of a matrix should allow you to do this.

It turns out to be almost as easy to prove this result for any two matrices $\mathbf{A}$ and $\mathbf{B}$ which are conformable for multiplication, so you might like to prove that more general result instead.

(iv) Prove that, if $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ symmetric matrices, then $\mathbf{AB}$ is symmetric if and only if $\mathbf{AB} = \mathbf{BA}$.

Recall that a matrix is called symmetric if it equals its transpose. This is another *if and only if* question, so you must prove the result in both directions.

Question 16

Let $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(i) The complex number $a + ib$ is represented by the matrix $a\mathbf{I} + b\mathbf{J}$. Show that if w and z are two complex numbers represented by the matrices $\mathbf{P}$ and $\mathbf{Q}$ respectively, then $w + z$ is represented by $\mathbf{P} + \mathbf{Q}$ and wz is represented by $\mathbf{PQ}$.

See the notes for the first part of question 10; this is identical except that $\mathbf{J}$ has been replaced by $-\mathbf{J}$.

(ii) The matrices $\mathbf{A}$ and $\mathbf{B}$ are $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$ respectively.

Express $\mathbf{A}$, $\mathbf{B}$, $\mathbf{B}^{-1}$ and $\mathbf{AB}^{-1}$ in the form $a\mathbf{I} + b\mathbf{J}$.

Write down the complex numbers represented by $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{AB}$ in the form $re^{\mathrm{i}\theta}$. Hence, or otherwise, show that

$$\arctan(\tfrac{1}{2}) + \arctan(\tfrac{1}{3}) = \tfrac{1}{4}\pi.$$

Once you have calculated $\mathbf{B}^{-1}$ and $\mathbf{AB}^{-1}$, then it should be straightforward to express them in the given form.

For the $re^{\mathrm{i}\theta}$ part, you will need to express the angles in terms of trigonometric ratios; this should give you enough to prove the final result. (Note that the final part asks about $\mathbf{AB}$, not $\mathbf{AB}^{-1}$!)

Question 17

Show that a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where $a_{11} \neq 0$ and $a_{11}a_{22} - a_{12}a_{21} \neq 0$, may be decomposed into the product of a *lower* triangular form matrix $\mathbf{L}$ and an *upper* triangular form matrix $\mathbf{U}$ such that $\mathbf{A} = \mathbf{LU}$ where

$$\mathbf{L} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & l & 0 \\ a_{31} & m & n \end{pmatrix} \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}.$$

Once you multiply $\mathbf{LU}$ explicitly, you will get enough equations to be able to find the values of l, m, n, p, q and r in terms of the elements of $\mathbf{A}$.

A system of simultaneous linear equations, $\mathbf{Ax} = \mathbf{c}$, may be solved by writing the equations as $\mathbf{LUx} = \mathbf{c}$ and letting $\mathbf{Ux} = \mathbf{u}$. The vector $\mathbf{u}$ is determined from $\mathbf{Lu} = \mathbf{c}$ and the solution $\mathbf{x}$ is then found from $\mathbf{Ux} = \mathbf{u}$. Since both $\mathbf{L}$ and $\mathbf{U}$ are triangular, these two sets of equations can be solved directly by back substitution.

Use this method to solve

$$\begin{aligned} x + y - z &= 2, \\ 3x + 2y + 5z &= 1, \\ 4x - y + 2z &= 0. \end{aligned}$$

Once you have expressed these simultaneous equations in the form $\mathbf{Ax} = \mathbf{c}$, you can use your above results to find $\mathbf{L}$ and $\mathbf{U}$ (and do check that $\mathbf{A} = \mathbf{LU}$).

Once you have done this, find $\mathbf{u}$ and then $\mathbf{x}$ following the instructions in the question.

Acknowledgements

The exam questions are reproduced by kind permission of Cambridge Assessment Group Archives. Unless otherwise noted, the questions are reproduced verbatim, except for some differences in convention between the papers as printed and STEP, specifically: the use of j to represent $\sqrt{-1}$ has been replaced by i; variables representing matrices are written in boldface type rather than italics; transpose is denoted $\mathbf{A}^{\mathrm{T}}$ rather than A'; derivatives and integrals are written using a roman d rather than an italic d.

In the list of sources below, the following abbreviations are used:

O&C	Oxford and Cambridge Schools Examination Board
SMP	School Mathematics Project
MEI	Mathematics in Education and Industry
QP	Question paper
Q	Question

1 UCLES, A level Mathematics, 1951, QP 190, Further Mathematics I, Q 2

2 UCLES, A level Mathematics, 1953, QP 188, Further Mathematics I, Q 2

3 UCLES, A level Mathematics, 1954, QP 188, Further Mathematics IV (Scholarship Paper), Q 1

4 UCLES, A level Mathematics, 1958, QP 437/1, Further Mathematics I, Q 1

5 O&C, A level Mathematics (SMP), 1968, QP SMP 34*, Mathematics II, Q 9; editorial change: clarify the last part of the question by adding the 'for all points P' part.

6 O&C, A level Mathematics (SMP), 1968, QP SMP 35, Mathematics III (Special Paper), Q 10; editorial changes here: the term 'order' is replaced by 'dimension'; the term 'frame of reference' has been replaced by 'coordinate system', and a diagram has been introduced to clarify the idea

7 O&C, A level Mathematics (MEI), 1969, QP MEI 32, Applied Mathematics III (Special Paper), Q 13; editorial changes: explained the dot notation, corrected a typographical error in the differential equation (missing dot) and made parentheses consistent

8 O&C, A level Mathematics (SMP), 1970, QP SMP 58, Further Mathematics IV, Q 6

9 UCLES, A level Mathematics, 1971, QP 842/0, Pure Mathematics (Special Paper), Q1

10 O&C, A level Mathematics (MEI), 1971, QP MEI 54, Pure Mathematics II, Q 4

11 UCLES, A level Mathematics, 1972, QP 848/0, Mathematics 0 (Special Paper), Q 12; editorial change: the matrix is called $\mathbf{A}$ rather than $\mathbf{a}$.

12 UCLES, A level Mathematics, 1972, QP 852/0, Further Mathematics 0 (Special Paper), Q 17

13 O&C, A level Mathematics (MEI), 1973, QP MEI 84, Pure Mathematics II, Q 4; editorial change: the row vector has have been replaced by a column vector

14 O&C, A level Mathematics (MEI), 1976, QP 128, Pure Mathematics I, Q 3

15 O&C, A level Mathematics (MEI), 1982, QP 9655/1, Pure Mathematics 1, Q 8

16 O&C, A level Mathematics (MEI), 1985, QP 9658/1, Further Mathematics 1, Q 1(a); editorial change: include a preliminary part showing that we can represent complex numbers in the form $a\mathbf{I} + b\mathbf{J}$ (and then remove the definitions of $\mathbf{I}$ and $\mathbf{J}$ from the main question)

17 O&C, A level Mathematics (MEI), 1988, QP 9658/0, Further Mathematics 0 (Special Paper), Q 9; editorial change: explain what the term 'back substitution' means in a footnote.