STEP Support Programme

STEP 3 Matrices Questions

This collection of questions is different from most of the STEP Support Programme, since matrices have not been on the STEP syllabus for many years. Note the following:

- These are **not** past STEP questions; they are from old A-level papers and similar.
- Many of these questions are *longer* or *shorter* than a typical STEP question.
- Many of these questions are *easier* or *harder* than a typical STEP question.
- The questions appear in chronological order of their origin, **not** in approximate order of difficulty.

The questions have been chosen to challenge you to think about matrices in a more sophisticated way than current A-level questions are likely to, and so will be a good preparation for what might appear on future STEP papers.

Matrices will first be examinable on STEP papers 2 and 3 from 2019 (under the new specification). There were a small number of STEP questions on the topic of matrices in the 1980s and 1990s. These can be found by searching for 'matrices' on the STEP database. Some of these are appropriate for today's STEP paper 2 or 3, but others require content beyond the current specification.

Acknowledgements (copyright details and question sources) for the questions in this module appear on the final page.

1 Express the determinant

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

as the product of factors which are linear in a, b, c.

Hence, or otherwise, find $x : y : z : u$ if

$$\begin{aligned} x + 2\ y + 3\ z + 4\ u &= 0, \\ x + 2^2 y + 3^2 z + 4^2 u &= 0, \\ x + 2^3 y + 3^3 z + 4^3 u &= 0. \end{aligned}$$

2 Prove that $(a - b)$ and $(x - y)$ are factors of the determinant

$$\begin{vmatrix} (a+x)^2 & (a+y)^2 & (a+z)^2 \\ (b+x)^2 & (b+y)^2 & (b+z)^2 \\ (c+x)^2 & (c+y)^2 & (c+z)^2 \end{vmatrix}$$

and factorise the determinant completely.

3 Given that α, β, γ are the roots of the equation

$$\begin{vmatrix} a & b & x \\ x & c & a \\ c & x & b \end{vmatrix} = 0,$$

prove that

$$\alpha^3 + \beta^3 + \gamma^3 = -6abc.$$

4 Prove that if the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{aligned}\right\}$$

are simultaneously satisfied by values of x, y, z which are not all zero, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Hence, or otherwise, eliminate x, y, z from the equations

$$a = \frac{x}{y-z}, \quad b = \frac{y}{z-x}, \quad c = \frac{z}{x-y}.$$

[Your answer should **not** be left in determinant form.]

5 A matrix $\mathbf{M}$ is said to be transposed into the matrix $\mathbf{M}^{\mathrm{T}}$ if the first row of $\mathbf{M}$ becomes the first column of $\mathbf{M}^{\mathrm{T}}$, the second row of $\mathbf{M}$ becomes the second column of $\mathbf{M}^{\mathrm{T}}$, and so on. Write down the transposes of the matrices

$$\mathbf{M} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{T} = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & c \\ a & 0 & 0 \end{pmatrix}.$$

Calculate the matrix products $\mathbf{M}^{\mathrm{T}}\mathbf{M}$ and $\mathbf{TM}$; show also that $(\mathbf{TM})^{\mathrm{T}} = \mathbf{M}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}$.

If the elements of $\mathbf{M}$ are the Cartesian coordinates of a point P, what information is provided by the element of $\mathbf{M}^{\mathrm{T}}\mathbf{M}$?

If the matrix $\mathbf{T}$ describes a transformation of the points P of three dimensional space, interpret geometrically the equation

$$(\mathbf{TM})^{\mathrm{T}}(\mathbf{TM}) = \mathbf{M}^{\mathrm{T}}\mathbf{M},$$

and find all appropriate values of a, b and c for which this equation holds for all points P.

6 The functions $t \to \mathbf{P}$, $t \to \mathbf{Q}$ map real numbers t onto matrices $\mathbf{P}$, $\mathbf{Q}$ of fixed dimension; that is, each element of each matrix is a real function of t, and we suppose all the functions to be differentiable. A scalar multiplier s is also a function of t. The derivative of a matrix is defined as a matrix whose elements are the derivatives of the elements of the original matrix. We write $\mathrm{d}\mathbf{P}/\mathrm{d}t$ as $\dot{\mathbf{P}}$, $\mathrm{d}s/\mathrm{d}t$ as $\dot{s}$, and so on.

(i) Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}(s\mathbf{P}) = \dot{s}\mathbf{P} + s\dot{\mathbf{P}}.$$

(ii) If the product $\mathbf{PQ}$ is defined, prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{PQ}) = \dot{\mathbf{P}}\mathbf{Q} + \mathbf{P}\dot{\mathbf{Q}}.$$

(iii) Prove that the derivative of a constant matrix is the zero matrix.

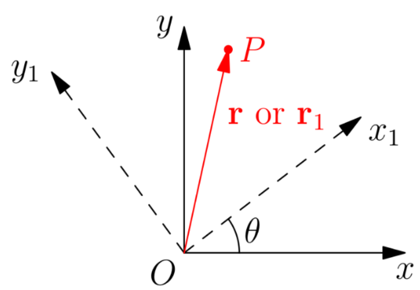
(iv) If $\mathbf{M}$ is the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

θ being a function of t, prove that $\dot{\mathbf{M}} = \mathbf{MJ}\dot{\theta}$, where

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The position of a particle in a plane is specified by a vector which can be described either as $\mathbf{r}$ relative to a coordinate system with rectangular axes Ox, Oy, or as $\mathbf{r}_1$ relative to a coordinate system with axes Ox_1, Oy_1, as shown in the diagram. The angle xOx_1, denoted by θ, is a function of t.



Write an equation connecting $\mathbf{r}$ with $\mathbf{r}_1$, and prove that

$$\dot{\mathbf{r}} = \mathbf{M}(\dot{\mathbf{r}}_1 + \mathbf{J}\mathbf{r}_1\dot{\theta}).$$

7 If

$$x = e^{kt}(a\cos\lambda t + b\sin\lambda t)$$

show that

$$\dot{x} = e^{kt}(a'\cos\lambda t + b'\sin\lambda t),$$

where

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} k & \lambda \\ -\lambda & k \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and dot denotes differentiation with respect to t, and find an expression for $\ddot{x}$ with coefficients given in a similar way.

A particular integral solution to

$$\ddot{x} + 2p\dot{x} + q^2 x = e^{kt}(C\cos\lambda t + D\sin\lambda t)$$

is

$$x = e^{kt}(a\cos\lambda t + b\sin\lambda t).$$

Show that

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} (k^2-\lambda^2) + 2pk + q^2 & 2k\lambda + 2p\lambda \\ -2k\lambda - 2p\lambda & (k^2-\lambda^2) + 2pk + q^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and hence write $\begin{pmatrix} a \\ b \end{pmatrix}$ in terms of $\begin{pmatrix} C \\ D \end{pmatrix}$.

Discuss any particular cases.

Note: The final part of this question is somewhat vague; a complete answer would not be expected on a STEP examination without a more explicit question.

8 Sketch the graph whose equation is

$$x^2 - y^2 = a^2 \quad (a > 0).$$

Prove that the point $P(a\cosh t, a\sinh t)$ lies on this graph for all real values of t; but that there are points of the graph which cannot be expressed in this form.

If U is the point of the graph for which $t = u$, find the area of the region $\mathscr{R}$ bounded by the curve, the line OU and the x-axis. (You may assume that $u > 0$.)

Prove that the transformation with matrix

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \qquad (\alpha \neq 0)$$

transforms the point P into another point of the curve. Into what region is $\mathscr{R}$ transformed by this? What is the area of the transformed region? Give reasons for your answers.

9 Show that the equations

$$\begin{aligned} 3x + 2y + z &= a - 1, \\ -2x + (a-2)y - az &= 2a, \\ 6x + ay + (a-2)z &= 3a - 6 \end{aligned}$$

have a solution, not necessarily unique, unless $a = \frac{2}{3}$.

Find the complete solution when $a = 0$ and when $a = 4$.

10 If z is the complex number $x + \mathrm{i}y$, $\mathrm{i} = \sqrt{-1}$, let $\mathbf{M}(z)$ denote the 2×2 matrix

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

Prove that

$$\mathbf{M}(z + z') = \mathbf{M}(z) + \mathbf{M}(z')$$

and that

$$\mathbf{M}(zz') = \mathbf{M}(z)\mathbf{M}(z').$$

Hence, or otherwise, show that

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}.$$

Hence find three real 2×2 matrices $\mathbf{A}$ such that $\mathbf{A}^3 = \mathbf{I}$ and a 2×2 matrix $\mathbf{B}$ such that $\mathbf{B}^2 + \mathbf{I} = \mathbf{B}$.

11 Using any method you wish, calculate the inverse $\mathbf{A}^{-1}$ of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 & 4 & 6 \end{pmatrix}.$$

Interpret geometrically in three dimensions the equations

$$\begin{aligned} x + y + 2z &= 4, \\ 2x + 3y + 3z &= 8, \\ 3x + 4y + \lambda z &= 7 + \lambda \end{aligned}$$

in the cases $\lambda = 6$ and $\lambda = 5$.

12 Using any method you prefer, and solving the problems in any order you prefer, find

(i) the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & 7 \\ 6 & 8 & 3 \end{pmatrix},$$

(ii) the solution of the simultaneous equations

$$\begin{aligned} x + 2y + 4z &= 3, \\ 3x + 5y + 7z &= 12, \\ 6x + 8y + 3z &= 31. \end{aligned}$$

The last of the three equations is replaced by

$$6x + 8y + az = b,$$

and it is found that the first two equations together with the new third one have more than one solution. Find a and b, and state a geometrical interpretation in three dimensions for these equations.

13 $\mathbf{A}$ is a 3×3 matrix whose elements are 0, 1 or -1 and each row and each column of $\mathbf{A}$ contains exactly one non-zero element. Prove that $\mathbf{A}^2$, $\mathbf{A}^3$, $\ldots$, $\mathbf{A}^n$ are all of the same form and deduce that $\mathbf{A}^h = \mathbf{I}$ for some positive integer $h \leqslant 48$.

Interpret the action of $\mathbf{A}$ on a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ geometrically.

Note: the final part of this question is somewhat vague and slightly beyond the requirements of the STEP 3 specification.

14 Find all possible solutions to

$$\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\1\\k\end{pmatrix} \qquad \text{(i)},$$

where

$$\mathbf{A}=\begin{pmatrix}1&2&-1\\2&1&1\\5&4&1\end{pmatrix},$$

stating explicitly the value of k that gives these solutions.

The equations

$$\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\1\\4\end{pmatrix} \quad\text{and}\quad \mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\1\\3\end{pmatrix}$$

may each be regarded as the equations of three planes; give a geometrical interpretation of your solution of equation (i) in terms of these sets of planes.

Write down the equation of the line, L, of intersection of the planes

$$x+2y-z=2 \quad\text{and}\quad 2x+y+z=1.$$

Find the equation of the plane through $(0,1,0)$ perpendicular to L and the coordinates of the points where this plane intersects the lines of intersection of

$$x+2y-z=2 \quad\text{and}\quad 5x+4y+z=3$$

and

$$2x+y+z=1 \quad\text{and}\quad 5x+4y+z=3.$$

15 (i) Given $\mathbf{M}=\begin{pmatrix}k&1\\0&k\end{pmatrix}$, calculate $\mathbf{M}^2$ and $\mathbf{M}^3$.

Suggest a form for $\mathbf{M}^n$ and confirm your suggestion, using the method of proof by induction.

(ii) Prove that, for any $n\times n$ matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{AB}=\mathbf{BA}$$

if and only if $(\mathbf{A}-k\mathbf{I})(\mathbf{B}-k\mathbf{I})=(\mathbf{B}-k\mathbf{I})(\mathbf{A}-k\mathbf{I})$ for all values of the real number k.

(iii) Prove that, for any $n\times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, $(\mathbf{AB})^{\mathrm{T}}=\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$, where $\mathbf{A}^{\mathrm{T}}$ is the transpose of $\mathbf{A}$.

(iv) Prove that, if $\mathbf{A}$ and $\mathbf{B}$ are $n\times n$ symmetric matrices, then $\mathbf{AB}$ is symmetric if and only if $\mathbf{AB}=\mathbf{BA}$.

16 Let $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(i) The complex number $a + \mathrm{i}b$ is represented by the matrix $a\mathbf{I} + b\mathbf{J}$. Show that if w and z are two complex numbers represented by the matrices $\mathbf{P}$ and $\mathbf{Q}$ respectively, then $w + z$ is represented by $\mathbf{P} + \mathbf{Q}$ and wz is represented by $\mathbf{PQ}$.

(ii) The matrices $\mathbf{A}$ and $\mathbf{B}$ are $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$ respectively.

Express $\mathbf{A}$, $\mathbf{B}$, $\mathbf{B}^{-1}$ and $\mathbf{AB}^{-1}$ in the form $a\mathbf{I} + b\mathbf{J}$.

Write down the complex numbers represented by $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{AB}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$. Hence, or otherwise, show that

$$\arctan(\tfrac{1}{2}) + \arctan(\tfrac{1}{3}) = \tfrac{1}{4}\pi.$$

17 Show that a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where $a_{11} \neq 0$ and $a_{11}a_{22} - a_{12}a_{21} \neq 0$, may be decomposed into the product of a *lower* triangular form matrix $\mathbf{L}$ and an *upper* triangular form matrix $\mathbf{U}$ such that $\mathbf{A} = \mathbf{LU}$ where

$$\mathbf{L} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & l & 0 \\ a_{31} & m & n \end{pmatrix} \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}.$$

A system of simultaneous linear equations, $\mathbf{Ax} = \mathbf{c}$, may be solved by writing the equations as $\mathbf{LUx} = \mathbf{c}$ and letting $\mathbf{Ux} = \mathbf{u}$. The vector $\mathbf{u}$ is determined from $\mathbf{Lu} = \mathbf{c}$ and the solution $\mathbf{x}$ is then found from $\mathbf{Ux} = \mathbf{u}$. Since both $\mathbf{L}$ and $\mathbf{U}$ are triangular, these two sets of equations can be solved directly by back substitution.[1]

Use this method to solve

$$\begin{aligned} x + y - z &= 2, \\ 3x + 2y + 5z &= 1, \\ 4x - y + 2z &= 0. \end{aligned}$$

[1] Back substitution: one of x, y and z appears on its own in an equation, so can be found immediately, then this can be substituted into another equation involving just this and one other unknown to find that second unknown, and these two can be substituted into the third equation to find the last unknown.

Acknowledgements

The exam questions are reproduced by kind permission of Cambridge Assessment Group Archives. Unless otherwise noted, the questions are reproduced verbatim, except for some differences in convention between the papers as printed and STEP, specifically: the use of j to represent $\sqrt{-1}$ has been replaced by i; variables representing matrices are written in boldface type rather than italics; transpose is denoted $\mathbf{A}^{\mathrm{T}}$ rather than A'; derivatives and integrals are written using a roman d rather than an italic d.

In the list of sources below, the following abbreviations are used:

O&C	Oxford and Cambridge Schools Examination Board
SMP	School Mathematics Project
MEI	Mathematics in Education and Industry
QP	Question paper
Q	Question

1 UCLES, A level Mathematics, 1951, QP 190, Further Mathematics I, Q 2

2 UCLES, A level Mathematics, 1953, QP 188, Further Mathematics I, Q 2

3 UCLES, A level Mathematics, 1954, QP 188, Further Mathematics IV (Scholarship Paper), Q 1

4 UCLES, A level Mathematics, 1958, QP 437/1, Further Mathematics I, Q 1

5 O&C, A level Mathematics (SMP), 1968, QP SMP 34*, Mathematics II, Q 9; editorial change: clarify the last part of the question by adding the 'for all points P' part.

6 O&C, A level Mathematics (SMP), 1968, QP SMP 35, Mathematics III (Special Paper), Q 10; editorial changes here: the term 'order' is replaced by 'dimension'; the term 'frame of reference' has been replaced by 'coordinate system', and a diagram has been introduced to clarify the idea

7 O&C, A level Mathematics (MEI), 1969, QP MEI 32, Applied Mathematics III (Special Paper), Q 13; editorial changes: explained the dot notation, corrected a typographical error in the differential equation (missing dot) and made parentheses consistent

8 O&C, A level Mathematics (SMP), 1970, QP SMP 58, Further Mathematics IV, Q 6

9 UCLES, A level Mathematics, 1971, QP 842/0, Pure Mathematics (Special Paper), Q1

10 O&C, A level Mathematics (MEI), 1971, QP MEI 54, Pure Mathematics II, Q 4

11 UCLES, A level Mathematics, 1972, QP 848/0, Mathematics 0 (Special Paper), Q 12; editorial change: the matrix is called $\mathbf{A}$ rather than $\mathbf{a}$.

12 UCLES, A level Mathematics, 1972, QP 852/0, Further Mathematics 0 (Special Paper), Q 17

13 O&C, A level Mathematics (MEI), 1973, QP MEI 84, Pure Mathematics II, Q 4; editorial change: the row vector has have been replaced by a column vector

14 O&C, A level Mathematics (MEI), 1976, QP 128, Pure Mathematics I, Q 3

15 O&C, A level Mathematics (MEI), 1982, QP 9655/1, Pure Mathematics 1, Q 8

16 O&C, A level Mathematics (MEI), 1985, QP 9658/1, Further Mathematics 1, Q 1(a); editorial change: include a preliminary part showing that we can represent complex numbers in the form $a\mathbf{I} + b\mathbf{J}$ (and then remove the definitions of $\mathbf{I}$ and $\mathbf{J}$ from the main question)

17 O&C, A level Mathematics (MEI), 1988, QP 9658/0, Further Mathematics 0 (Special Paper), Q 9; editorial change: explain what the term 'back substitution' means in a footnote.